Dofinansowane przez Unię Europejską

PROPER

PROBABILITY AROUND US
PROBABILITY FOR EVERYONE

A methodical guide for teacher

Probability

PROPER
PROBABILITY AROUND US
PROBABILITY FOR EVERYONE

PROPER

PROBABILITY AROUND US
PROBABILITY FOR EVERYONE

Co-funded by
the European Union

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

A methodical guide for teachers of mathematics in secondary school

Probability with Stochastic Graphs

Ireneusz Krech, Pavel Tlustý

PROPER
PROBABILITY AROUND US
PROBABILITY FOR EVERYONE

Co-funded by the European Union

1 Stochastic graphs vs. in-school probability theory teaching process

1.1 Probability theory vs. intuition

Mathematical research and discovery is not only a result of one's pure deduction, inductive thinking and analogy-based reasoning but it is also a result of intuitive thinking (see [31]). The formal approach towards mathematics is often opposed to the intuitive approach. Abstractions and schemas are contrasted to "seeing" and "perception" of general, important mathematical constructions and quantitative and space relations. The inspiration and beginning of all discoveries as well as the point that gives certainty in all kinds of reasoning and the author of new ideas, hypotheses or statements is "obviousness", "common sense", that is – intuition.

For a long time Freudenthal used to replace the word "intuition" with a phrase "shaping of mathematical objects" (see [7]). He was doing so because of a wide range of meanings that the word "intuition" has in different languages. Freudenthal also wrote (see [8]) that "intuitions without concepts are empty, and concepts without intuitions are blind".

Stochastic intuitions are the ability of drawing judgments and beliefs of probabilistic character without any conscious inference or even without perceiving the clues which justify that belief or judgement. It is an ability allowing us to estimate properly the probabilistic characteristics (the event's probability, the expected value, distribution or stochastic independence) of a given sample or population on the basis of incomplete data about the sample and without any (conscious) reasoning or analysis, when the estimation is based only on one's experience or knowledge.

The intuitive conclusions are the ones which we consider obvious, we draw them instantly, almost without thinking, without any reasoning, calculations or argumentations on the basis of images, schemes or situation models that we have in memory. Intuitive thinking is thinking about an abstract situation through its specific model (see [24]).

In [35], [36] and [37] we can find the research of psychologists A. Tversky and D. Kahneman which show that people do not have their probabilistic intuitions properly developed. Humans were not provided even with basic probabilistic intuitions through evolution.

Wrong probabilistic intuitions may be mathematically – based. They can be a result of lack of basic probabilistic, stochastic and combinatorial knowledge, but they can also rise from its poor acquisition (a formalized lecture does not eliminate mistakes in intuitive judgments). They can also have psychological background. A formal explanation of the probability theory and statistics rules is not enough to eliminate those "incorrect representations" in the process of probabilistic predicting, which is seen as an important pre-decisive process by psychologists. The psychological research show that in the process of predicting people do not use probabilistic arguments as much as they use some rules, principles and strategies.

Tversky and Kahneman analyzed the basis of incorrect representations (incorrect intuitions) in situations concerning probability estimations. They point out the divergence between a subjective probability (i.e. estimation of probability given by a person as his /her estimation of a chance of a given event to happen) and objective, normative probability resulting from a probabilistic model. They conducted the research as a part of a bigger project concerning problems of teaching mathematics. They studied the strategies used by people of different age and occupation while solving specific stochastic (combinatorial, as a matter of fact) problems.

J. M. Shaughenessy's research shows how vast is the role of personal contact between a person and empiricism (drawing lots, working with statistical data, using the predevelopped data, like the results of chance games, calculating frequencies, confronting the a posteriori judgments with the ones made a priori) in developing correct stochastic intuitions which appear in using heuristic strategies properly. The same research proves that teaching probabilistic theory in too formalized way, apart from statistics, omitting the empirical aspect of probabilistic issues and leaving out some classical paradoxes like problems – stochastic surprises does, not remove incorrect intuitions. Tversky and Kahneman emphasize the fact, that the same mistakes are made by "stochastically naive" students (the ones with no probabilistic experience) and adults – even ones who had graduated from advanced but formalized stochastic courses. They find mistakes of this kind made even by psychologists who have some knowledge of stochastics.

1.2 The functional teaching of mathematics

The idea of functional teaching is the basic strategy of didactically correct process of teaching-learning mathematics. It may also be seen as a basic strategy of discovering

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

and creating mathematics by students (see [34]). It is a universal method, recommended in teaching different subjects, but in mathematics – because of an abstract and operative character of mathematical notions – it has got a particular meaning. In functional teaching we try to show mathematics from the notional side, not through the algorithms and rules, as it was in the mechanistic approach. The definitions, rules, reasoning or theorems are important, but they come later on, as a summary, a result of different activities, discovering and using algorithms. According to the integral approach, mathematics should grow from reality, everyday situations. In the functional method the objects and phenomena of the students' environment do not have to be the starting point of mathematical issues. Along with real situations we can use the ones artificially created, using special teaching aids as well as purely abstract problems. The care for precision and order, for clarity and understanding of mathematical issues, for the compatibility of school and scientific notions is vital in the functional teaching. The basis of the student's mathematical activity is his awareness of where in the "math construction" he actually is at the moment. The overriding aim of this teaching method is the student gaining operative knowledge not on the basis of chaotic trials of solving schematic problems or too "casual work", but through the student's activities carefully planned by the teacher. Only a well trained teacher, with a good knowledge of methodology can plan the student's work properly and lead the student to create sequent elements of mathematical knowledge, stressing "mathematical activity, working in mathematical world and its connection to reality, creative experience gathered by the student gradually through solving problems open for creativity at his level" (see [24]).

Through the functional teaching the constructive approach is accomplished. The student creates his own knowledge integrated with various materials and tasks, on the way of reach experience gathered in cooperation with the teacher and fellow students. However, it is not about the superficial shaping of mathematical issues leading to the answer to "what is it" question. It is about active study of techniques and methods that allow the student to solve "the how do we construct" problems. We can find the confirmation of this idea in Piaget's Where does education aim in an extended and supported by numerous research form. Piaget claims there that the basic condition of the whole mind shaping process, which is especially important in the matters that lead young learners to science, is using active methods of teaching. They allow the student to spontaneously search for solutions and demand each truth that is to be discovered to be rediscovered by the student and not only passed to him.

1.3 Probability versus stochastic games

Probability is present at every stage of teaching math teachers. But they often lack proper tools of introducing probability at school. This situation is eve a bigger challenge for primary and secondary school teachers. A real didactic suggestion is to introduce stochastic issues on the grounds of chance games that are often followed by lots of stochastic paradoxes. Solving different problems connected to those games leads to

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

proper understanding of elementary characteristics and acquiring correct intuitions. Thanks to the paradoxes occurring in those games we can set didactic situations leading to didactic reflections both tor students and teachers. Although probability is present on the elementary and secondary stage of education of math teachers, mathematicians often lack specific tools for teaching probability. Even well trained math teachers, having broad knowledge of mathematics, usually need some additional professional training connected to teaching probability. General rules of teaching which are usually effective in other branches of mathematics are not necessarily as effective in teaching probability theory. This situation is even a greater challenge for primary school teachers. Although teachers do not need a very high level of mathematical knowledge, it is necessary for them to understand the basic notions of mathematics they teach at schools thoroughly, including deep understanding of relations and connections among different aspects of that knowledge (see [25]). The additional elements that are important in the professional teachers' knowledge are described in [1]:

a) epistemology: a reflection on meanings of different notions, like different meanings of probability (see [2]);

b) learning: foreseeing problems in the student's learning, mistakes, obstacles and strategies;

c) didactical means and methods: experience in good selection of examples and didactic situations; ability to analyze the textbooks, curricula and other documents critically; ability to adapt the statistics to different levels of education;

d) ability to engage the students in work and make them interested in what they do; taking their beliefs and attitudes into consideration;

e) interactions: ability to create effective communication in the classroom and using rating as a means of instructing students.

Classical paradoxes play a great role in teaching probability. Because of them we can organize some didactical activities for the math teachers. The aim of these activities is to provoke their reflection on the basic probabilistic notions. These activities also help the teachers understand the students' obstacles and difficulties in understanding probability and they allow them to expand their own methodological and didactical base.

Introduction of the stochastic graph into the probability teaching process is to create, develop and shape those correct stochastic intuitions in a proper way. Simultaneously, we build this process by introducing a specific kind of chance experiments and problems generated by them.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

1.4 Penney's game and a stochastic graph

There are two possible results of a coin toss. We shall code them in such a way:
o — the result will be heads and r -– the result will be tails. We shall call the r result a success and the o result a failure. The result of k coin tosses, which is a k-arrangement of $\{o, r\}$ set we shall call a series of successes and failures, in short – a series of k length.

Let a and b be a defined series of successes and failures of k length. Repeating a coin toss as many times as needed to get k trial result make the a or b series is called waiting for the a or b series and marked as δ_{a-b}. Let us connect the events of:
A={waiting δ_{a-b} will finish with the a series},
B={waiting δ_{a-b} will finish with the b series}
with the δ_{a-b} chance experiment.

Let us mark the A event as $\{\dots a\}$ and its probability as $P(\dots a)$. The B event shall be marked as $\{\dots b\}$ and its probability as $P(\dots b)$.

In a short article [27] Walter Penney discusses repeating a coin toss as many times as needed to get three times heads or a heads-tails-heads series. Let $\delta_{ooo-oro}$ mean the described chance experiment. Penney suggests a lot game for two players. In the game the $\delta_{ooo-oro}$ experiment is conducted (it is not important who tosses the coin). One of the players wins if the experiment ends up with the ooo result, and the other player wins when the experiment ends with the oro result. The game described above we shall call $g_{ooo-oro}$. The fact that the ooo and oro series are equally possible to happen would suggest that the game is fair. But the probability that the waiting $\delta_{ooo-oro}$ will end up with the oro series is 0,6, while the probability that it will end up with the ooo series is 0,4. Penney finds the probabilities on a way of particular reasoning (see [29], p. 415) and he does not try to hide his being surprised by the fact that the game is not fair. The oro series gives the player a bigger chance to win that the ooo one. This is the interpretation of the results and the calculation on the real-life ground. So the oro series is called better than the ooo one.

The problem of the fairness of chance games in case of waiting for other pairs of series of heads and tails – those are called Penney's games – the issues connected to the paradox characteristics of the success-failure series in waiting for one of them to occur, as well as the problem of time needed for such waiting (meant as a period of time taken by the game, when time is measured with the number of coin tosses executed) are called Penney's problems in mathematical literature. Only in case of some pairs of heads and tails series the Penney's game is fair. Such series are called equally good.

Some of the results of research on the Penney's problems are gathered in [11] monograph and [32], [13], [14], [15], [16], [17], [21] and [22] articles.

A tool for examining the countable probabilistic spaces for waitings for success-failure series is a stochastic graph. Such a waiting for a series of successes and failures is a chance experiment of a random number of stages.

The research on the probabilistic space for the waiting for one of many success-failure series may be brought down to searching for the probabilities of reaching each of the absorbing levels. Waiting for a success-failure series is often interpreted as a

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

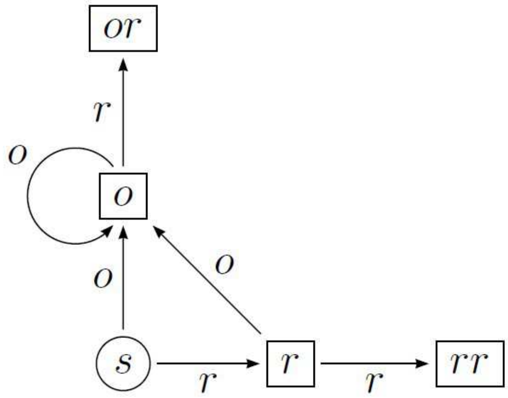

Fig. 1: Stochastic graph - game g_{rr-or}

homogeneous Markov's chain with the non-empty set of absorbing stages (see [19]) and it is suggested to use an iconic representation, along with the algebraical one, that is a stochastic graph. Traditionally, such calculations are based on sequences and differential equations. The essence of argumentations based on the stochastic graph is, among others, a reduction of cycles and loops on the graph (we call them reductions of the graph), or transition from a graph with unlimited number of passages to a limited-passage graph (see [18]). It is a development of methods and tools suggested long ago by Arthur Engel in [3], [4] and [5] (see also [20]).

The stages of a homogeneous Markov's chain can be interpreted as points on a plain and called knots. The knot that represents the beginning stage is called starting knot. Each knot representing an absorbing stage is called edge knot. If the probability of getting from a j stage to a k stage in one step is positive, then we connect those knots with the oriented subsection of a line or curve and we mark that subsection k. We call that subsection an arc. A graph constructed in such a way is an iconic representation of a Markov's chain.

At the beginning (before conducting the first stage of experiment) we place a pawn in the starting knot of the graph. If a stage ends with the j result we move the pawn along the j arc. The route of the pawn ends when it gets to an edge knot, that is at the rim of the graph (see [23]). Picture 1 shows a stochastic graph being a board of the g_{or-rr} game.

If the pawn gets to the o knot at any stage of the game, it is certain (the probability equals 1) that it will get to the knot (finish) or – that is the player waiting for this series wins. For the pawn getting to the o knot the heads must be the result of first or second toss, so the probability of this event is 0,5+0,25=0,75. The pawn gets to the rr knot only if the first and second toss result with tails, then the other player wins, and this happens with the probability of 0,25.

It is just one example of elementary, simple, but very elegant and making a great impression reasoning based on a stochastic graph. There are lots of such examples can

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

be found in the quoted literature.

A natural generalization of discussed problems is replacing a coin toss with any chance experiment having two possible results of non-equal probability (that is a Bernoulli's trial) or a chance experiment having more than two results. Then we can discuss the series of successes and failures or series of colors (flags).

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

2 Probability in Probability Spaces Connected with Generalised Penney's Games

2.1 Discrete probability space and probabiliy in such a space

Let Ω be an arbitrary at least two-element and at most countable set. A non-negative function $p : \Omega \to R$ which fulfils the condition

$$\sum_{\omega\in\Omega} p(\omega) = 1,$$

is called a probability distribution on the set Ω.

Let $\mathcal{Z} = 2^{\Omega}$. Define the function P on the set $\mathcal{Z}$ in the following way:

$$P(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ p(\omega), & \text{if } A = \{\omega\}, \\ \sum\limits_{\omega\in A} p(\omega), & \text{if } A \text{ is a set with at least two elements,} \end{cases}$$

It is not difficult to show that the function P fulfils the conditions of the axiomatic definition of probability. Therefore the triple $(\Omega, \mathcal{Z}, P)$ is a probability space. It is called a discrete probability space due to the cardinality of Ω.

The elements of the family $\mathcal{Z}$ are called events and the value of the function P for a set A from the family $\mathcal{Z}$ is called the probability of event A. In order to define a discrete probability space $(\Omega, \mathcal{Z}, P)$ it is necessary and sufficient to define a probability distribution p on Ω. For this reason the pair (Ω, p) may also be called a descrete probability space. In the following considerations the construction of a probability space will be understood as the construction of a pair (Ω, p) in which Ω is a set containing at least two elements and at most countable and p is a probability distribution on Ω.

2.2 Series of successes and failures, waiting for one of the two series and its probability model

A random experiment with two possible results is called a Bernoulli trial or a trial if the probabilities of these two results are positive. Let one of them be denoted by 1 and called a success while the other is denoted by 0 and called a failure. Let us also denote the probabilities of the success and failure by u and v respectively. Therefore $0 < u < 1$ i $u + v = 1$.

Every result of the experiment in which a particular Bernoulli trial is performed m times (i.e. the result of a Bernoulli scheme of m trials) is called a series of successes and failures. Number m is called the length of the series. A series of successes and failures of length m will be represented as the m-arrangement of the set $\{0, 1\}$.

Let a and b be fixed series of successes and failures of the length m. The random experiment of repeating the given trial until the results of the last m trials create series a or series b is called waiting for one of the two series a or b and is denoted by d_{a-b}.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

Let Ω_{a-b} be the set of such arrangements of $\{0, 1\}$ with at most m terms in which the last m terms create the series a or b, while neither the series a nor the series b is created by any previous subsequence of consecutive m terms. The set Ω_{a-b} consists of all results of the random experiment d_{a-b}. For $\omega \in \Omega_{a-b}$ let $j(\omega)$ stand for the number of the terms of the sequence ω which are equal to 1. Let the symbol $|\omega|$ denote the length of the sequence ω, i.e. the number of its terms. Define the function p_{a-b} on Ω_{a-b} by the following formula:

$$p_{a-b}(\omega) = u^{j(\omega)} \cdot v^{|\omega|-j(\omega)} \text{ dla } \omega \in \Omega_{a-b}.$$

The function p_{a-b} is a probability distribution on the set Ω_{a-b}, so the pair (Ω_{a-b}, p_{a-b}) is a probability space. It is called the probability model of the random experiment d_{a-b}. The set Ω_{a-b} is not finite but it is countable, the pair (Ω_{a-b}, p_{a-b}) is an infinite (countable) probability space.

2.3 Some generalisation of Penney's game onto a series of successes and failures

Let a i b be fixed series of successes and failures of length m. Two players G_a and G_b take part in the game. A particular Bernoulli trial is repeated until the results of last m trials create the series a - in which case the player G_a wins - or the series b, which means that the player G_b is a winner. Let us denote the game described above by g_{a-b}. It is a generalisation of the game suggested in 1969 by Walter Penney for $u = \frac{1}{2}$ (see W. Penney, Problem 95: Penney-Ante, Journal of Recreational Mathematics 7-1974, p. 321).

In this game the random experiment d_{a-b} is performed, modelled by the probability space (Ω_{a-b}, p_{a-b}). Let $\{a \prec b\}$ denote the event $\{$the series a appears before the series $b\}$ and let $P(a \prec b)$ stand for its probability.

2.4 Stochastic graph and probability space induced by it

While repeating the trials it is necessary to continuously control the result of the last m trials in order to decide whether the game is over and who is the winner. This procedure may be rationalised by interpreting the course of the experiment d_{a-b} as wandering of a pawn on a stochastic graph. This interpretation refers to the idea of simulation of the course of homogeneous Markov chains presented by Arthur Engel in [?].

Waiting for one of the series of successes and failures is a homogeneous Markov chain. Let us consider the stochastic graph of this Markov chain. Let Ω^* be the set of all paths on this graph. To each path let us assign the product of numbers related to the consecutive edges of this path. This product is called the weight of the path. The function which to each path assigns its weight will be denoted by p^*. The function p^* is a probability distribution on Ω^*, so the pair (Ω^*, p^*) is a probability space. It is called the space induced by the stochastic graph. All the subsequent calculations and reasonings are conducted in such a probability space induced by a stochastic graph.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

If (Ω_{a-b}, p_{a-b}) is the probability model of the random experiment d_{a-b} defined above and (Ω^*, p^*) is a probability space induced by the stochastic graph of the random experiment d_{a-b}, then both spaces are isomorphic.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

3 Faster and equally fast series of successes and failures

Many factors have the influence on discovering and understanding mathematics, among others intuition. The abstraction and the schematics in teaching mathematics are being confronted with the vision and perceiving of general, essentially important mathematical structures and the quantitative and spatial relations. Our common sense i.e. our intuition is the author of any ideas, statements or hypotheses, it is he inspiration, the beginning of any discovery and the clue delivering us confidence in reasoning o any type. In the chapter the examples of stochastic paradoxes are presented. These paradoxes are connected with special relations defined in a set of successes and failures series, that standing against our intuitions appear to be a mean of the mathematical activation.

Let $u \in (0,1)$, $\Omega_{0-1} = \{0,1\}$, $p^u_{0-1}(1) = u$ and $p^u_{0-1}(0) = 1-u$. Any experiment which model (see [30]) is a probabilistic space $(\Omega_{0-1}, p^u_{0-1})$ is called Bernoulli trial or briefly a trial and is denoted by δ^u_{0-1}. The result denoted by number 1 is called success, and the result denoted by number 0 is called failure. The number u is called the probability of success.

In this work as a model of a probabilistic many-stage experiment we assume a probabilistic space created with the following rules of stochastic tree:

(R1) the result of a many-stage random experiment δ as an element of the set Ω_δ, at the same time so-called as elementary event is represented by a sequence of results of subsequent stages,

(R2) the probability distribution p_δ on the set Ω_δ we define by so-called multiply rule that says: if $\omega \in \Omega_\delta$ and $\omega = (a_1, a_2, \ldots, a_n)$, than the pair (Ω_k, p_k) is the model of the k-th stage and $a_k \in \Omega_k$ for $k = 1, 2, \ldots, n$, so

$$p_\delta(\omega) = p_1(a_1) \cdot p_2(a_2) \cdot \ldots \cdot p_n(a_n).$$

Let $m \in \mathbb{N}_1$. Any result of m-times repeated Bernoulli trial e.i. any arrangement of m out of 2 elements (of the set $\{0,1\}$) is called the series of successes and failures and is denoted by α. The number m is called the length of the series α.

Let α be any stated series of successes and failures of length m, where $m \in \mathbb{N}_1$. Repeating the Bernoulli trial as long as results of m last trials will create the series α is called the waiting for series α and is denoted by δ^u_α. The number of trials done in an experiment δ^u_α is called the waiting time for series α. This number (mentioned before beginning of awaiting) is a random variable in a probabilistic model of awaiting δ^u_α and it is denoted by T^u_α. The number $E(T^u_\alpha)$ or expected value of the random variable T^u_α - is an average waiting time for series α.

Definition. Let α_1 and α_2 be the series of successes and failures. If

$$E(T^u_{\alpha_1}) = E(T^u_{\alpha_2}),$$

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

so the series α_1 and α_2 are called equally fast at the point u and are denoted by

$$(\alpha_1 \Diamond \alpha_2)_u.$$

Definition. Let α_1 and α_2 be the series of successes and failures. If

$$E(T^u_{\alpha_1}) < E(T^u_{\alpha_2}),$$

so the series α_1 is called faster than series α_2 at the point u and is denoted by

$$(\alpha_1 \lhd \alpha_2)_u.$$

Let α_1, α_2 be the stated series of successes and failures of length m_1 and m_2 respectively. Repeating the Bernoulli trial as long as:

– the results of m_1 last trials create the series α_1,

– or the results of m_2 last trials create the series α_2,

is called the waiting for one of 2 series of successes and failures and is denoted by $\delta^u_{\alpha_1-\alpha_2}$.

Let us introduce an event in a probabilistic model of the experiment $\delta^u_{\alpha_1-\alpha_2}$:

$$A_j = \{\text{waiting } \delta^u_{\alpha_1-\alpha_2} \text{ will be finished by series } \alpha_j\} = \{\ldots\alpha_j\} \quad \text{for } j = 1, 2.$$

The probability of the event A_j is denoted by $P^u_{\alpha_1-\alpha_2}(\ldots\alpha_j)$.

Definition. Let us consider the waiting $\delta^u_{\alpha_1-\alpha_2}$. If

$$P^u_{\alpha_1-\alpha_2}(\ldots\alpha_1) = P^u_{\alpha_1-\alpha_2}(\ldots\alpha_2),$$

so the series α_1 and α_2 are called equally well at the point u and are denoted by

$$(\alpha_1 \approx \alpha_2)_u.$$

Definition. Let us consider the waiting $\delta^u_{\alpha_1-\alpha_2}$. If

$$P^u_{\alpha_1-\alpha_2}(\ldots\alpha_1) > P^u_{\alpha_1-\alpha_2}(\ldots\alpha_2),$$

so the series α_1 is called better than series α_2 at point u and is denoted by the symbol

$$(\alpha_1 \gg \alpha_2)_u.$$

The below examples illustrate paradoxical properties of relations:

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

$$\approx, \gg, \lhd \text{ and } \Diamond.$$

Example 1. Let us consider a series of successes and failures: $10, 01$ and 00 for $u = \frac{1}{2}$. Here we have

$$(10 \approx 01)_{\frac{1}{2}} \wedge (01 \approx 00)_{\frac{1}{2}} \wedge (10 \gg 00)_{\frac{1}{2}}.$$

Therefore the relation $\approx$ is not a transitive relation in a set of successes and failures (at stated parameter $u = \frac{1}{2}$).

Example 2. Let us consider a series of successes and failures: $1101, 1011$ and 0111 for $u = \frac{1}{2}$. Here we have

$$(1101 \gg 1011)_{\frac{1}{2}} \wedge (1011 \gg 0111)_{\frac{1}{2}} \wedge (0111 \gg 1101)_{\frac{1}{2}},$$

therefore in these three series, no one is best (e.i. better than any of the two other). Relation $\gg$ in a set of successes and failures, is not a transitive relation.

Example 3. Let $\alpha_1 = 0111$, $\alpha_2 = 1110$. Here we have

$$(0111 \Diamond 1110)_{\frac{1}{2}},$$

but in the waiting $\delta^{\frac{1}{2}}_{0111-1110}$ there is

$$(0111 \gg 1110)_{\frac{1}{2}}.$$

From the fact that series are equally fast it doesn't result that they are equally good. At the same time we notice that $E(T^{\frac{1}{2}}_{0111}) = E(T^{\frac{1}{2}}_{0111}) = 16$ and the average waiting time for one of these two series 0111 and 1110 or the average duration time of the experiment $\delta^{\frac{1}{2}}_{0111-1110}$ is 14.25.

Example 4. Let $\alpha_1 = 1111$, $\alpha_2 = 1110$. Here we have

$$(1110 \lhd 1111)_{\frac{1}{2}},$$

however in the waiting $\delta^{\frac{1}{2}}_{1111-1110}$ we have

$$(1110 \approx 1111)_{\frac{1}{2}}.$$

It doesn't appear from the fact that series are faster that they are better.

Example 5. Let $\alpha_1 = 111$, $\alpha_2 = 0011$. Here we have

$$(111 \lhd 0011)_{\frac{1}{2}},$$

but in the waiting $\delta^{\frac{1}{2}}_{111-0011}$ we have

$$(0011 \gg 111)_{\frac{1}{2}}.$$

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

Faster series can be "worse" series.

Example 6. Let $\alpha_1 = 1100$, $\alpha_2 = 000$. In the waiting $\delta^{\frac{1}{2}}_{111-0011}$ we have

$$(1100 \gg 000)_{\frac{1}{2}}$$

but

$$(000 \lhd 1100)_{\frac{1}{2}}.$$

Better series don't need to be faster series.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

4 Waiting time for series of successes and failures and fairness of random games

Definition 1.
Let $u \in (0,1)$ is an arbitrary real number. We call the random trial modelled with its sample space $(\Omega_{0-1}, p^u_{0-1})$, where

$$\Omega_{0-1} = \{0,1\}, \qquad p^u_{0-1}(1) = u \quad \text{and} \quad p^u_{0-1}(0) = 1-u,$$

the Bernoulli trial and we denote it with δ^u_{0-1}. The results of the Bernoulli trial are denoted with 0 resp. 1 and we call them failure resp. success.

Definition 2.
We denote each result of m-multiple repetition of the trial δ^u_{0-1} with α which is called a series of successes and failures with the length of m. We say that the series of α_1 is a subseries of series of α_2, and we write $\alpha_1 \subset \alpha_2$, if α_1 as a string of 0 and 1 is a subseries of series α_2. If α_1 is not a subseries of α_2, we write $\alpha_1 \not\subset \alpha_2$. If $\alpha_1 \not\subset \alpha_2$ and $\alpha_2 \not\subset \alpha_1$, we say that series α_1 and α_2 are differential.

Definition 3.
Let α is the chosen series of successes and failures of length m. We call the repetition of trial δ^u_{0-1} till the results m of the last trial make series α the waiting for series α and we denote it with δ^u_α. The number of repetitions of trial δ^u_α is a random variable T^u_α on set $\Omega_{T^u_\alpha} = \{m, m+1, m+2, \ldots\}$. Number $E(T^u_\alpha)$ is the mean waiting time for series α.

Definition 4.
Let α_1 and α_2 are series of successes and failures. If $E(T^u_{\alpha_1}) = E(T^u_{\alpha_2})$ then we call series α_1 and α_2 of the same speed at point u.

If $E(T^u_{\alpha_1}) < E(T^u_{\alpha_2})$, than we call series α_1 faster than series α_2 point u.

Definition 5.
Given two differential series α_1 and α_2 of successes and failures, $|\alpha_j| = m_j$ for $j = 1,2$. We repeat trial δ^u_{0-1} as long as:

– results m_1 of the last trials make series α_1,

– or results m_2 of the last trials make series α_2,

is called the waiting for one of two series of successes and failures, and we denote it with $\delta^u_{\alpha_1-\alpha_2}$ and its probability model with $(\Omega_{\alpha_1-\alpha_2}, p^u_{\alpha_1-\alpha_2})$.

If

$$P^u_{\alpha_1-\alpha_2}(\ldots\alpha_1) = P^u_{\alpha_1-\alpha_2}(\ldots\alpha_2),$$

we call series α_1 and α_2 alike at point u. If

$$P^u_{\alpha_1-\alpha_2}(\ldots\alpha_1) > P^u_{\alpha_1-\alpha_2}(\ldots\alpha_2),$$

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

then we call series α_1 better than series α_2 at point u.

□ Given the following random game. Player G_A tosses a coin till the results of the last two tosses make series lr. Player G_B tosses a coin till the results of the last two tosses make series ll. The winner is the player who tosses first his/her series. Which of the player has a higher chance to win?

Let us denote the tails with 0 and the heads with 1. It is possible to simulate the game course with the rambling of a stone in a stochastic graph. (see [3] and [30]) in Fig. 1.

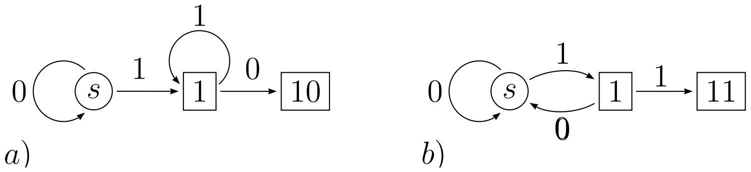

Fig. 1

Using the algorithm for the calculation of the mean of the rambling time in the stochastic graph (see [30], p. 399), we get

$$E(T_{10}) = 4 \quad \text{and} \quad E(T_{11}) = 6,$$

which means that series 10 is quicker than series 11, and so there is a higher chance to win for player G_A.

□ Given another game. Players G_A and G_B toss a coin till the results of the last two tosses make series lr (G_A wins) or series ll (G_B wins). This game represents the waiting δ^u_{10-11}, with its graph in Fig. 2.

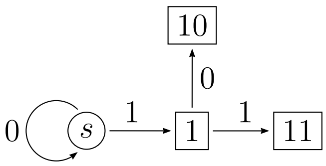

Fig. 2

It seems that if series 10 is quicker than series 11, it must be also better. Paradoxically series 10 and 11 are alike, which is obvious from the symmetry of stochastic graph in Fig. 2.

This begs the question: What random variable connected with the waiting time for series 10 and 11 do we have to think about to be able to make a decision which of the series is better based on its mean?

In Fig. 3, there is a modified graph δ^u_{10-11}.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

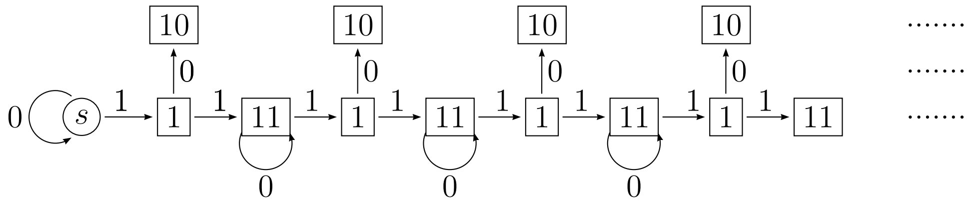

Fig. 3

If we calculate the waiting time for series 10 and we get combination 11, then we start again from node $\boxed{11}$, which becomes the starting node. We obtain a stochastic graph with infinite number of nodes. The time of rambling in such a graph is a random variable T_{10}^{10-11}, which represents also the waiting time for series 10. Fig. 4 shows several consecutive periods of this modifications.

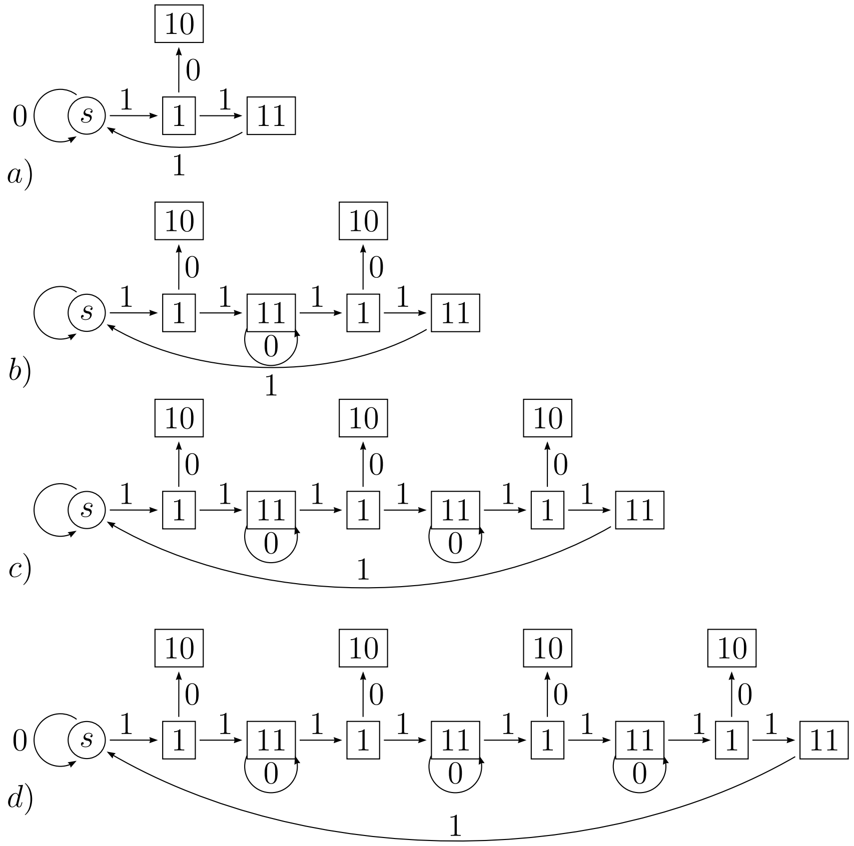

Fig. 4

Mean $E(T_{10}^{10-11})$ of the waiting time in the graph in Fig. 4 can be determined as a limit of a sequence. In the same way, we determine $E(T_{11}^{10-11})$. As

$$E(T_{10}^{10-11}) = E(T_{11}^{10-11}) = 6,$$

the both series are alike.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

5 A stochastic graph as a specific tool of mathematics and argumentation

Let us consider a random board game of g_{x-y}. The board consists of two circles: o_v and o_w. At the beginning inside the o_v circle there are x coins and inside the o_w circle there are y coins. Let us assume that $x + y = 3$ and $y \neq 3$. There are two players, G_a and G_b, in the game. They take turns and toss the 3 coins placed on the game board. Coins that show heads stay in the circle they were originally placed in. Coins that show tails change their circle. If all the coins end up in the ow circle after the toss, the player who tossed them wins. Let us assume the G_a player takes the first run (see [6]).

Later in the article we will answer the question: which of the g_{3-0}, g_{2-1} and g_{1-2} games is best (meaning which gives the best chance to win) for the G_a player and which is best for the G_b player.

We will mark the experiment conducted in the game as δ_{x-y}. Let A_{x-y} mean the G_a player wins and B_{x-y} mean the G_b player wins.

The random experiment of δ_{x-y} is conducted in some phases. Each single phase consists of a coins toss and placing them in the circles. The experiment status after the nth phase is a pair of (v_n, w_n), where v_n means the number of coins placed in the o_v circle and w_n means the number of coins placed in the o_w circle after this phase. As $v_n + w_n = 3$, the experiment status after the nth phase is defined by the v_n number. The possibilities here make a set of $S = \{0, 1, 2, 3\}$. We can interpret them as the graph loops. The beginning of the game becomes the start loop and the experiment status before the game (the 0 stage) becomes the edge loop (see[1]). Let us mark the probability of the experiment going from the j to the k status as p_{jk}. If $p_{jk} > 0$, we connect the j and k loop points on the graph with a line. Then we write the p_{jk} number next to the line. This way we get a stochastic graph and a game board simultaneously.

I. Let us start with the g_{3-0} game. The graph and the stages of constructing it are shown in picture 1.

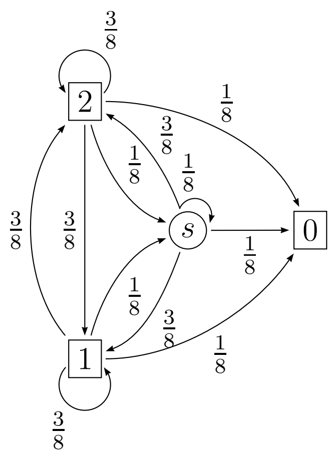

Fig. 1.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

This graph is a particularly useful tool of argumentation while calculating the probability of a certain player's victory in the game.

We call the $0, 1$ and 2 phases the inner ones. We can notice that once the δ_{3-0} experiment gets to one of the inner stages, the next toss will lead it either to the 0 state, with the probability of $\frac{1}{8}$, or to another inner state, with the probability of $\frac{7}{8}$. These symmetries prove that the graph from picture 1 reduces to the one from picture 2.

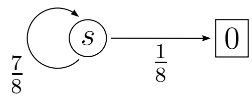

Fig. 2.

The course of the game and its result can be registered if we include the time it takes. Picture 3 shows the graph of the δ_{3-0} experiment after this modification.

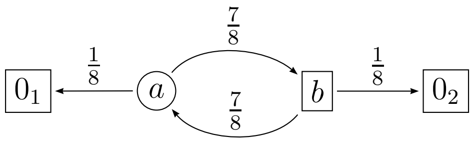

Fig. 3.

The G_a player can win only if the experiment after an even toss takes the $\boxed{0}$ stage of the graph from picture 2 or - which is really the same - the $\boxed{0_1}$ stage of the graph from picture 3. The G_b player can win if the experiment after an odd toss takes the $\boxed{0}$ stage of the graph from picture 2 or - which is really the same - the $\boxed{0_2}$ stage of the graph from picture 3. From the interpretations above we can see that:

1) In case of the graph from picture 2 there is

$$P(A_{3-0}) = \frac{1}{8} + (\frac{7}{8})^2 \cdot \frac{1}{8} + (\frac{7}{8})^4 \cdot \frac{1}{8} + \ldots = \frac{\frac{1}{8}}{1-(\frac{7}{8})^2} = \frac{8}{15} \quad \text{and} \quad P(B_{3-0}) = \frac{7}{15}.$$

2) Let $P(A_{3-0}) = x$ and $P(B_{3-0}) = y = 1 - x$. We know from the graph shown in picture 3, that:

$$x = \frac{1}{8} + x \cdot (\frac{7}{8})^2, \quad \text{so} \quad x = \frac{8}{15} \quad \text{and} \quad y = \frac{7}{15}.$$

So the player who starts the game has better chance to win it.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

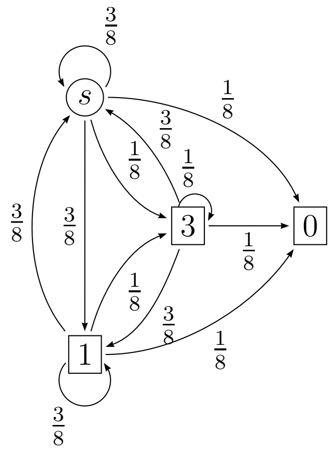

Fig. 4.

II. Let us consider the g_{2-1} game. The δ_{2-1} experiment status at the beginning of the game is 2. Picture 4 shows the stochastic graph of the δ_{2-1} experiment.

Just like in the previous game of g_{3-0}, once the experiment δ_{2-1} gets to one of the inner stages, the next coins toss leads it either to the 0 stage, with the probability of $\frac{1}{8}$ (the player who started the game wins it), or to another inner stage, with the probability of $\frac{7}{8}$. So we can see that the graph of this experiment is isomorphic with the one from picture 2 (and, considering the time, with the graph from picture 3), and

$$P(A_{2-1}) = \frac{8}{15} \quad \text{and} \quad P(B_{2-1}) = \frac{7}{15}.$$

III. It is easy to see that when we consider the g_{1-2} game we get a graph (as its board) isomorphic with the graphs of the g_{3-0} and g_{2-1} games. So that

$$P(A_{1-2}) = \frac{8}{15} \quad \text{and} \quad P(B_{1-2}) = \frac{7}{15}.$$

Finally we get:

$$P(A_{3-0}) = P(A_{2-1}) = P(A_{1-2}) = \frac{8}{15}$$

and

$$P(B_{3-0}) = P(B_{2-1}) = P(B_{1-2}) = \frac{7}{15}.$$

Summary. The final conclusion of our deliberations is surprising: the players' chance to win does not depend on the experiment status at the beginning of the game, the player who starts the game wins it with the probability of $\frac{8}{15}$ and the player who takes the second turn tossing the coins wins it with the probability of $\frac{7}{15}$.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

6 Stochastic tools for calculating sums of certain numerical series

The mathematical analysis theorems let decide if a given numerical series is convergent but in general they don't let appoint its sum. The work concerns stochastic tools for calculating sums of certain series.

If A is a finite set , then $\overline{\overline{A}}$ denotes the cardinality of the set A, that is the number of its elements. In this work the notation $\mathbb{N}_k$ denotes the set $\{k, k+1, k+2, \ldots\}$.

6.1 Numerical series and its sum

Definition 1 Let us assume that $(a_n)_{n\in\mathbb{N}_1}$ is an infinite numerical sequence. Let $s_1 = a_1$ and let $s_n = a_1 + a_2 + a_3 + \cdots a_n$ for $n = 2, 3, 4, \ldots$. The number s_n is called the n-th partial sum of the sequence $(a_n)_{n\in\mathbb{N}_1}$ and an infinite sequence $(s_n)_{n\in\mathbb{N}_1}$ is called the series and is denoted by $\sum a_n$. If the sequence $(s_n)_{n\in\mathbb{N}_1}$ has a finite limit, then we say that the series $\sum a_n$ is convergent. If $\lim\limits_{n\to\infty} s_n = s$, then number s is called the sum of the sequence and is denoted by

$$\sum_{n=1}^{\infty} a_n = s.$$

This sum s is understood as an infinite sum $a_1 + a_2 + a_3 + \cdots$ that is as a sum of all infinitely numerous terms of the sequence $(a_n)_{n\in\mathbb{N}_1}$ that is

$$a_1 + a_2 + a_3 + \cdots = \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n).$$

Definition 2 We say that the series $\sum a_n$ is absolutely summable if the series $\sum |a_n|$ is convergent.

Theorem 3 If the series $\sum |a_n|$ is convergent, then the series $\sum a_n$ is also convergent, in other words: any absolutely summable series is convergent (see [10]).

Definition 4 Let $a_n = a \cdot q^{n-1}$ for $n \in \mathbb{N}_1$, where a and q are established real numbers and $q \neq 0$. The series $\sum a \cdot q^{n-1}$, as a series created on a base of a geometrical sequence $(a \cdot q^{n-1})_{n\in\mathbb{N}_1}$, is called the geometrical series.

Theorem 5 If $|q| < 1$, then the geometrical series $\sum a \cdot q^{n-1}$ is convergent and

$$\sum_{n=1}^{\infty} a \cdot q^{n-1} = \tfrac{a}{1-q}.$$

The fraction $\frac{a}{1-q}$ is then the sum of all (infinite number of) terms of geometrical sequence $(a \cdot q^{n-1})_{n\in\mathbb{N}_1}$, if $|q| < 1$.

Example 6 If $u \in (0, 1)$, then the sequence $(a_n)_{n\in\mathbb{N}_1}$, where $a_n = (1-u)^{n-1} \cdot u$ for $n \in \mathbb{N}_1$ is a geometrical sequence, for which $a = u$ and $q = 1 - u$, therefore

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n = \tfrac{u}{1-(1-u)} = \tfrac{u}{u} = 1.$$

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

6.2 Notions of the discrete probability calculus and concept of series

Definition 7 Let Ω be any set of at least two elements and let it be a countable set at most. The nonnegative function $p : \Omega \to R$ satisfying the condition

$$\sum_{\omega:\omega\in\Omega} p(\omega) = 1,$$

is called the probability distribution on the set Ω.

Let $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$. If p is the probability distribution on a set Ω and $p(\omega_n) = p_n$ for $n \in \mathbb{N}_1$, then the sum of the series $\sum p_n$ is equal to 1.

Example 8 The geometrical sequence $(a_n)_{n\in\mathbb{N}_1}$, where $a_n = (1-u)^{n-1} \cdot u$ and $0<u<1$ is the probability distribution on the set $\mathbb{N}_1$. It is called the geometrical distribution.

In order to determine discrete probability space $(\Omega, \mathcal{Z}, P)$ it is needed an sufficient to determine the probability distribution p on the set Ω, so we can treat a discrete probability space as a pair (Ω, p).

If the pair (Ω, p) is the probability space, Ω is a countable set and A is a countable subset of the set Ω, then $P(A)$ is the sum of a certain series. Probability of such event A, that is a sum of a certain series, may be calculated with the use of certain stochastic theorems.

Definition 9 Let (Ω, p) be a probability space. Any function $X : \Omega \longrightarrow \mathbb{R}$ is called a random variable in this space (Ω, p).

Definition 10 Let $\Omega_X = X(\Omega)$ and $\{X = x_j\} = \{\omega \in \Omega : X(\omega) = x_j\}$ for $x_j \in \Omega_X$. The set $\{X = x_j\}$ is an event in the probability space $(\Omega, \mathcal{Z}, P)$ created by the pair (Ω, p). Let $P(X = x_j)$ denotes its probability. The function $p_X : \Omega_X \longrightarrow \mathbb{R}$ defined by the formula

$$p_X(x_j) = P(X = x_j) \text{ for } x_j \in \Omega_X,$$

is called the distribution of the random variable X.

Theorem 11 The function p_X is the probability distribution on the set Ω_X, therefore the pair (Ω_X, p_X) is the discrete probability space.

Example 12 We say that the random variable X has a geometrical distribution, if

$$\Omega_X = \mathbb{N}_1 \text{ and } p_X(n) = (1-u)^{n-1} \cdot u \text{ for } n \in \mathbb{N}_1, \text{ where } 0 < u < 1.$$

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

Definition 13 Let Ω_X be a set of values and p_X a distribution of a random variable X in a discrete probability space (Ω, p). The expected value of the random variable X is called the number

$$E(X) = \sum_{x_j : x_j \in \Omega_X} x_j \cdot p_X(x_j),$$

under the condition that if Ω_X is a countable set, the series $\sum x_j \cdot p_X(x_j)$ is absolutely summable.

Theorem 14 If $X_1, X_2, X_3, \ldots, X_n$ are the random variables in the same discrete probability space (Ω, p) and each of them has its expected value, then the expected value has also the random variable $X_1 + X_2 + X_3 + \cdots + X_n$ and

$$E(X_1 + X_2 + X_3 + \cdots + X_n) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_n).$$

6.3 Duration of random experiment consisting of random number of stages as the duration of random walks on a graph and its average duration versus series.

Among random experiments consisting of stages we set apart experiments with random number of stages. We assume that successive stages are carried out in successive time units. According to this convention, the number of stages becomes the duration of a given experiment.

Let $0 < u < 1$. The Bernoulli trial is a random experiment δ^u_{0-1} whose outcome is random and can be either of two possible outcomes: success denoted by number 1 or failure denoted by number 0. Number u is the probability of success.

Example 15 There are two results of a toss-up (flipping a coin):
— `it came down heads`, the result is coded by 0 and called failure,
— `it came down tails`, the result is coded by 1 and called success.
A toss-up is a Bernoulli trial. Probability of success $u = \frac{1}{2}$.

Definition 16 Repeating the attempt δ^u_{0-1}, as long as we get success k times, is an experiment of random number of stages, which is called awaiting on k successes, or the Pascal process and it is denoted by δ^u_k. In the case $k = 1$ this is awaiting for the first success.

Definition 17 Let $\alpha \in \{0,1\}^k$. The sequence α is interpreted as a result of k-th repetition of the attempt δ^u_{0-1} and it is called a series of successes and failures. The number k is called the series length and is denoted by $|\alpha|$.

Definition 18 Let $\alpha \in \{0,1\}^k$. Repeating the attempt δ^u_{0-1}, as long as the results of k last stages will create the series α is called awaiting for series α and is denoted by δ^u_α.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

Number of repetitions of the attempt δ_{0-1}^u until such time as the series α is gained, is a random variable T_α, which is called the awaiting time for series α.

A random experiment of random number of stages is a uniform Markov chain and has its own Engel stochastic graph. The course of such experiment is interpreted as a random walk across the mentioned graph. There are presented two algorisms in [30] (p. 398-400): the absorption algorism and the algorism of average random walking time across the stochastic graph. First of these algorisms is used for calculating probability with which a pawn walking across the graph will reach the established node point. The other from algorisms lets calculate average random walking time of the pawn across the stochastic graph. In both algorisms the bottom line is to solve the set of linear equations in which number of equations is equal to the number of nodes in this stochastic graph.

Example 19 Let $0 < u < 1$. Let us consider a numerical sequence $(a_n)_{n\in\mathbb{N}_1}$, where $a_n = n \cdot u \cdot (1-u)^{n-1}$ for $n \in \mathbb{N}_1$ and the series $\sum a_n$. In areas of mathematical analysis the sum of this series may be found with the use of the differential theorem for function series. In areas of probability calculus the calculation of the sum of this series may be reduced to solution of simple set of linear equations.

Let T_1^u be the time of awaiting for the first success. The distribution of this random variable is a geometrical sequence $(b_n)_{n\in\mathbb{N}_1}$, where

$$b_n = P(T_1^u{=}n) = u \cdot (1-u)^{n-1} \quad \text{for} \quad n \in \mathbb{N}_1.$$

The course of awaiting δ_1^u may be interpreted as a random walking across a stochastic graph presented in Figure 1. In this interpretation the random variable T_1^u is a time of random walking across this graph.

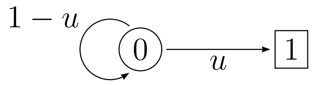

Fig. 1.

The expected value of the random variable T_1^u is the sum of the series $\sum n \cdot u \cdot (1-u)^{n-1}$, where $n \in \mathbb{N}_1$. Let e_j denotes the average random walking time across the graph in Figure 1, that started in the node j in this graph (j is the wait state, $j = 0$ or $j = 1$). Therefore it is $E(T_1^u) = e_0$. Using the algorism of average random walking time across the stochastic graph we receive the system of equations:

$$\begin{cases} e_0 = 1 + (1-u) \cdot e_0 + u \cdot e_1, \\ e_1 = 0, \end{cases}$$

that is $e_0 = \frac{1}{u}$, and therefore

$$\sum_{k=1}^{\infty} k \cdot u \cdot (1-u)^{k-1} = \frac{1}{u}.$$

Then it is $E(T_1^u) = \frac{1}{u}$.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

Example 20 Let T_k^u be the time of awaiting for k successes (or the duration time of Pascal process). The distribution of this random variable T_k^u is e sequence $(p_n)_{n\in\mathbb{N}_k}$, where

$$p_n = P(T_k^u = n) = \binom{n-1}{k-1} \cdot u^k \cdot (1-u)^{n-k} \quad \text{for} \quad n = k, k+1, k+2, \ldots.$$

The expected value of the random variable T_k^u is the sum of the series $\sum n \cdot p_n$, where $n \in \mathbb{N}_k$, that is

$$E(T_k^u) = \sum_{n=k}^{\infty} n \cdot \binom{n-1}{k-1} \cdot u^k \cdot (1-u)^{n-k}. \tag{6.3.1}$$

Let us consider the Pascal process for $k = 5$. In the Figure 2 we have a stochastic graph for this awaiting for five successes.

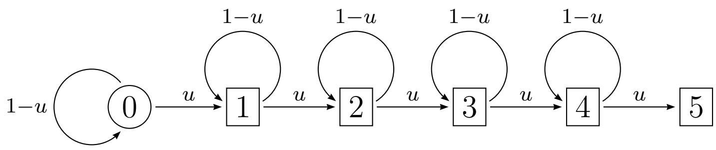

Fig. 2.

1° Let $T_{j\to k}^u$ be an awaiting time for success number k in situation when we already have j successes. Let $k = j+1$ for $j = 0, 1, 2, 3, \ldots, k-1$. The random variable $T_{j\to(j+1)}^u$ is the awaiting time for the first success, therefore $E(T_{j\to(j+1)}^u) = \frac{1}{u}$ and

$$T_k^u = T_{0\to1}^u + T_{1\to2}^u + T_{2\to3}^u + T_{3\to4}^u + \cdots + T_{(k-1)\to k}^u,$$

and thus - it results from the theorem 3, that

$$E(T_k^u) = E(T_{0\to1}^u) + E(T_{1\to2}^u) + E(T_{2\to3}^u) + \cdots + E(T_{(k-1)\to k}^u) = k \cdot \tfrac{1}{u} = \tfrac{k}{u},$$

and taking into account (6.3.1) we get that

$$\sum_{n=k}^{\infty} n \cdot \binom{n-1}{k-1} \cdot u^k \cdot (1-u)^{n-k} = k \cdot \frac{1}{u} = \frac{k}{u}. \tag{6.3.2}$$

2° It may be checked that the formula (6.3.2) may be also obtained by using algorism of the random walking average time across the stochastic graph of awaiting for k successes.

Example 21 Let

$$A_n = \{(a_1, a_2, a_3, \ldots, a_n) \in \{1, 2, \ldots, n\}^n : a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots \leqslant a_n\}.$$

If $a \in A_n$ and $a = (a_1, a_2, \ldots, a_n)$, then $\overline{a} = a_1 \cdot a_2 \cdot \ldots \cdot a_n$. Let us consider the series $\sum n \cdot \frac{4!}{5^4} \cdot \left(\sum_{a\in A_{n-5}} \overline{a}\right)$ for $n \in \mathbb{N}_5$ and its sum

$$\sum_{n=5}^{\infty} n \cdot \frac{4!}{5^4} \cdot \left(\sum_{a\in A_{n-5}} \overline{a}\right). \tag{6.3.3}$$

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

Let $U_{1\to 5}$ denotes an urn with five balls numbered from 1 to 5. The experiment of a random number of stages is the randomization (drawing balls) with returning them to the urn, as long as each ball is drawn at least once. Such an experiment is called the collector's scheme. The number of a drawn ball will be called the drawn number. In the Figure 3 we have the stochastic graph of this scheme. The node label is the state of the collection, that is the number of already drawn numbers. The number of drawings (randomizations) in this experiment is the random variable $T_{1\to s}$. This is the duration time of this collector's scheme.

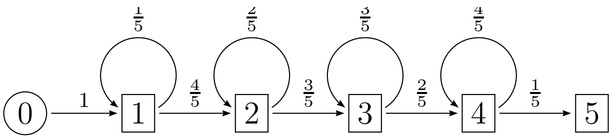

Fig. 3.

The expected value of the duration time of this collector's scheme, that is $E(T_{1\to s})$ is the sum

$5 \cdot \frac{4!}{5^4} + 6 \cdot \frac{4!}{5^5}(1+2+3+4) + 7 \cdot \frac{4!}{5^6} \cdot (1\cdot 1 + 1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4$
$+3\cdot 3 + 3\cdot 4 + 4\cdot 4) + \cdots$.

It may be checked that the last finite sum is the sum of the form (6.3.3).

It results from the theorem 4 that the expected value of the duration time of the discussed collector's scheme (see Figure 3) is the following sum $1 + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + 5$, that is $E(T_{1\to s}) = \frac{137}{12}$ and therefore

$$\sum_{n=5}^{\infty} n \cdot \frac{4!}{5^4} \cdot \left(\sum_{a\in A_{n-5}} \overline{a} \right) = \frac{137}{12}.$$

Definition 22 The sequence $(f_1, f_2, f_3, \ldots)$, where $f_1 = f_2 = 1$ and $f_n = f_{n-2} + f_{n-1}$ for $n \in \mathbb{N}_3$ is called the Fibonacci sequence. There is f_3=2, f_4=3, f_5=5, f_6=8 etc. The number f_n is called the n-th Fibonacci number.

Example 23 Let us consider the numerical series $\sum \frac{f_{n-1}}{2^n}$, where $n \in \mathbb{N}_2$ and its sum

$$\sum_{n=2}^{\infty} \frac{f_{n-1}}{2^n} = \frac{1}{2^2} + \frac{1}{2^3} + \frac{2}{2^4} + \frac{3}{2^5} + \frac{5}{2^6} + \frac{8}{2^7} + \cdots ,$$

where the numbers $f_1, f_2, f_3, \ldots$ are the successive terms of Fibonacci sequence.

In this context, let us consider the repetitions of toss-up until the tails appear two times in a row. Such a random experiment δ_{11} is called awaiting for the series 11 (this means the series of heads or tails). Its result is at least two element sequence consisting of elements from the set $\{0, 1\}$ and satisfying the condition that the last two element create the series 11 and non of two previous successive elements create such series. By Ω_{11} we denote the set of random experiment results δ_{11}. Therefore it may be for example: $01011 \in \Omega_{11}$, $000011 \in \Omega_{11}$. If ω is the result of awaiting δ_{11}, then by $|\omega|$ we denote its length, this is the number of elements in the sequence ω. Let

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

$$p_{11}(\omega) = \left(\tfrac{1}{2}\right)^{|\omega|} \text{ dla } \omega \in \Omega_{11}.$$

The function p_{11} is the probability distribution on the set Ω_{11}, therefore the pair (Ω_{11}, p_{11}) is a probability space. This is a model of awaiting δ_{11}.

Let us connect an event

$$B_n = \{\text{the series 11 will be gained after } n\text{-th toss-up}\} \text{ for } n \in \mathbb{N}_2$$

with the random experiment δ_{11}. In the probability space (Ω_{11}, p_{11}) there is

$$B_2 = \{11\}, B_3 = \{011\}, B_4 = \{0011, 1011\}, B_5 = \{10011, 01011, 00011\}.$$

The power of event connected with awaiting δ_{11} is the number of results (of the experiment δ_{11}) that favour this event. Let $b_n = \overline{\overline{B_n}}$. There is $b_2 = 1$, $b_3 = 1$, $b_4 = 2$, $b_5 = 3$ etc. Let us notice that $b_n = f_{n-1}$ for $n = 2, 3, 4, \ldots$. All the results of awaiting δ_{11}, except the result 11, are the sequences ending with the series 011. For $n = 6, 7, 8, \ldots$ the event B_n is favoured by the number of results equal to the number of $(n-3)$ terms arrangements with repetitions of the set $\{0, 1\}$, in which the digit 1 does not appear two times in a row. All the results favouring the event B_n are the sequences of the length n, therefore

$$p_{11}(\omega) = \left(\tfrac{1}{2}\right)^n \text{ for any } \omega \in B_n.$$

It results from the definition 4 that

$$P(B_n) = b_n \cdot \left(\tfrac{1}{2}\right)^n = f_{n-1} \cdot (\tfrac{1}{n})^n = \tfrac{f_{n-1}}{2^n} \quad \text{for} \quad n = 2, 3, 4, \ldots \quad .$$

There is $\bigcup\limits_{n=2}^{\infty} B_n = \Omega_{11}$ and $B_j \cap B_k = \emptyset$ for $j \neq k$, therefore the events $B_2, B_3, B_4, \ldots$ create a partition of Ω_{11}. It results from here that $\sum\limits_{n=2}^{\infty} P(B_n) = 1$, that is

$$\sum_{n=2}^{\infty} P(B_n) = \sum_{n=2}^{\infty} \frac{f_{n-1}}{2^n} = 1. \tag{6.3.4}$$

Let us consider the series $\sum \frac{f_{n-1}}{2^n}$, where $n \in \mathbb{N}_3$, and its sum

$$\sum_{n=3}^{\infty} \tfrac{f_{n-1}}{2^n} = \tfrac{1}{2^3} + \tfrac{2}{2^4} + \tfrac{3}{2^5} + \tfrac{5}{2^6} + \tfrac{8}{2^7} + \cdots \quad . \tag{6.3.5}$$

From the formula (6.3.4) it results that

$$\sum_{n=3}^{\infty} \tfrac{f_{n-1}}{2^n} = \sum_{n=2}^{\infty} \tfrac{f_{n-1}}{2^n} - \tfrac{f_1}{2^2} = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

The sum of the last series may be found in the other way, basing on the stochastic approach.

In a game in which two players G_A and G_B are participating, a toss-up is being repeated until: after tails up two times heads up will appear one by one (...100) and then the G_A player is winning, or after two heads up one by one, the tails up will appear (...001) and then the G_B player is winning. It is a special case of so-called Penney's game. It isn't important here who is flipping a coin.

The random experiment carried out in this game is an awaiting for one of two series of heads and tails. Let us denote this by $\delta_{100-001}$. The set $\Omega_{100-001}$ of its results

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

is a set of at least three-elements sequences consisting of elements from the set $\{0, 1\}$ and satisfying the condition that last three elements create the series 100 or series 001 and none three previous successive elements create these series. For example the results of the experiment $\delta_{100-001}$ are the sequences: 00001, 1010100, 10101010100. Let $p_{100-001}$ denotes the function that associates to each result of experiment $\delta_{100-001}$ its probability. If $\omega \in \Omega_{100-001}$ and $|\omega| = n$, then $p_{100-001}(\omega) = (\frac{1}{2})^n$ for $n = 3, 4, 5, \dots$. The pair $(\Omega_{100-001}, p_{100-001})$ is the probability space.

Let us consider the events:
A=$\{$the experiment $\delta_{100-001}$ will close with series 100$\}$.
A_n=$\{$the experiment $\delta_{100-001}$ will close with series 100 after n attempts$\}$.
In the described game the player G_A is winning whenever the event A happens. Each result containing event A is the sequence ending with series 100, in which no three successive previous elements create series 100, nor series 001. It results from here that no two successive previous elements create series 00. If c_n denotes the number of results realizing the event A_n, then

$$c_3 = 1,\ c_4 = 2 \text{ and } c_n = f_{n-1} \text{ for } n \in \mathbb{N}_5.$$

There is

$$A = \bigcup_{n=3}^{\infty} A_n \text{ and } P(A_n) = \frac{f_{n-1}}{2^n}, \text{ for } n \in \mathbb{N}_3,$$

that is

$$P(A) = \sum_{n=3}^{\infty} \frac{f_{n-1}}{2^n}.$$

The probability of the event A is then the sum (6.3.5) that is the sum of the discussed series.

The course of the random experiment $\delta_{100-001}$ (that is the course of the game) is interpreted as random walk of pawn across the graph in the Figure 4 (this is a kind of draught-board to the discussed game).

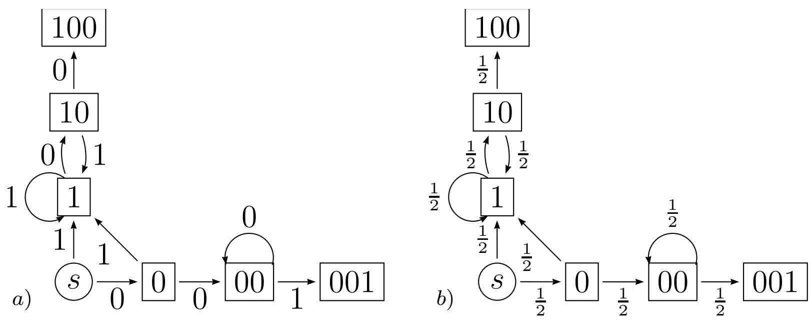

Fig. 4.

We put the pawn in the node start and after the subsequent toss-up we move it along the line corresponding to the result of the toss-up[1] . The experiment (and the game) is ending when the pawn reaches the node $\boxed{100}$ (an event A takes place and

[1] In the Figure 4a there are digits 0 and 1 assigned to the arrows in the graph, as results of a toss-up, in the Figure 4b instead of these results their probabilities are assigned to the arrows.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

player G_A is winning) or to the node $\boxed{001}$ (player G_B is winning). If the pawn finds its way to the node $\boxed{00}$, in fact the game becomes already decided (player G_B is winning). If the pawn reaches the node $\boxed{1}$, the game is also decided (player G_A is winning). It results from here that $P(A)$ is a probability of reaching the node $\boxed{1}$ by the pawn. The pawn will reach the node $\boxed{1}$, when at first stage tails fall out or when at first stage heads will fall out but in second stage - tails, what means that the probability of reaching the node $\boxed{1}$ is equal to $\frac{1}{2}+\frac{1}{4}$ that is $P(A)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$. Therefore we have

$$P(A)=\sum_{n=3}^{\infty}\frac{f_{n-1}}{2^n}$$

and at the same time $P(A)=\frac{3}{4}$, that is

$$\sum_{n=3}^{\infty}\frac{f_{n-1}}{2^n}=\frac{3}{4}.$$

Example 24 Let us return to the random experiment δ_{11} and let us consider the series $\sum n\cdot f_{n-1}\cdot(\frac{1}{2})^n$, where $n\in\mathbb{N}_2$ and its sum

$$\sum_{n=2}^{\infty}n\cdot f_{n-1}\cdot\left(\tfrac{1}{2}\right)^n=2\cdot 1\cdot\left(\tfrac{1}{2}\right)^2+3\cdot 1\cdot\left(\tfrac{1}{2}\right)^3+4\cdot 2\cdot\left(\tfrac{1}{2}\right)^4+\cdots\ .$$

Let T_{11} be an awaiting time for the series 11 measured by the number of toss-up attempts carried out in the experiment δ_{11}. Let us return to the event B_n discussed in Example 8. Let us notice that $\{T_{11}=n\}=B_n$ for $n\in\mathbb{N}_2$, and thus

$$P(T_{11}=n)=P(B_n)=f_{n-1}\cdot(\tfrac{1}{2})^n\quad\text{for}\quad n\in\mathbb{N}_2.$$

The expected value of the random variable T_{11} is by definition the sum of series $\sum n{\cdot}P(T_{11}=n)$, where $n\in\mathbb{N}_2$ that is the sum

$$\sum_{n=2}^{\infty}n\cdot f_{n-1}\cdot(\tfrac{1}{2})^n,$$

where numbers $f_1, f_2, f_3, \ldots$ are the subsequent elements of Fibonacci sequence.

The Figure 5 presents the graph of awaiting for two tails one after another. The course of this awaiting is interpreted as random walking of the pawn across this graph. The duration time of the experiment δ_{11} is measured by the number of arrows travelled by the pawn from the start start to the finish $\boxed{11}$. In this interpretation the awaiting time for series 11 is the time of random walking around the graph in Figure 5.

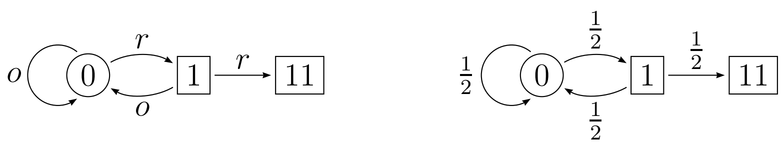

Fig. 5.

Using the algorism of average random walking time around the stochastic graph we get that $E(T_{11})=6$. From one side $E(T_{11})$ is the sum $\sum_{n=2}^{\infty}n\cdot f_{n-1}\cdot\left(\frac{1}{2}\right)^n$, but from the other site $E(T_{11})=6$ and thus

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

$$\sum_{n=2}^{\infty} n \cdot f_{n-1} \cdot \left(\frac{1}{2}\right)^n = 6.$$

6.4 The arithmetic-geometric series and stochastic

Definition 25 Let $a_n = a + (n-1)r$ and $g_n = bq^{n-1}$ for $n \in \mathbb{N}_1$. The sequence (a_n) is an arithmetic sequence and the sequence (g_n) is geometric sequence. Let us consider a new sequence (c_n), where $c_n = a_n \cdot g_n$ for $n \in \mathbb{N}_1$. The series created on the basis of the sequence (c_n) that is the series $\sum[a + (n-1)r] \cdot b \cdot q^{n-1}$, where $n \in \mathbb{N}_1$ is called the arithmetic-geometric series.

It may be proved that if $|q| < 1$, then the series $\sum[a+(n-1)r] \cdot b \cdot q^{n-1}$ is convergent. Finding its sum is a relatively complicated task of mathematical analysis. It may be proved (see [?]) that

$$a \cdot b + (a + r) \cdot bq + (a + 2r) \cdot bq^2 + (a + 3r) \cdot bq^3 + \cdots = \frac{a \cdot b}{1-q} + \frac{r \cdot bq}{(1-q)^2}. \quad (6.4.1)$$

We will show how to find the sum of the above series using the tools of stochastic.

It occurs for the sum of the arithmetic-geometric series $\sum[a + (n-1)r] \cdot b \cdot q^{n-1}$ as the sum of all infinite number of elements

$$\begin{gathered} ab + (a + r)bq + (a + 2r)bq^2 + (a + 3r)bq^3 + (a + 4r)q^4 + \cdots = \\ ab + abq + abq^2 + abq^3 + abq^4 + \cdots + brq + 2brq^2 + 3brq^3 + 4brq^4 + \cdots = \\ ab(1 + q + q^2 + q^3 + q^4 + \cdots) + brq(1 + 2q + 3q^2 + 4q^3 + \cdots). \end{gathered}$$

The sum in the first bracket of the last expression is a sum of all infinite number of geometrical sequence elements, that is

$$ab(1 + q + q^2 + q^3 + q^4 + \cdots) = \tfrac{ab}{1-q}.$$

The sum in the second bracket of this expression that is the infinite sum

$$1 + 2q + 3q^2 + 4q^3 + \cdots$$

will be found using the fact the expected value of the random variable T of geometrical distribution $P(T = n) = q^{n-1} \cdot (1-q)$, where $|q| < 1$ and $n \in \mathbb{N}_1$ is the ratio $\frac{1}{1-q}$. Because we have the identity

$$1 + 2q + 3q^2 + 4q^3 + \cdots = \tfrac{E(T)}{1-q} = \tfrac{1}{(1-q)^2}.$$

We proved in this way (and what's more - using the stochastic tools) that the formula (6.4.1) is the formula for the sum of the arithmetic-geometric series.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

7 Stochastic Tree and Construction of Discrete Probability Spaces and Series Summation

The presented chapter deals with sums of certain series which are interpreted in context of discrete probability space as consecutive ball draws performed in phases. Stochastic trees represent an unusual tool for determining sums of numerical series.

7.1 Discrete random experiment

Definition 26 Discrete random experiment is called a real or artificial ([?], p. 16-17 and also [?], p. 13-14) experiment δ, which develops and results in accordance with the following conditions: 1) set Ω_δ of all results of the experiment is as much countable as possible, 2) for each result, it is possible to determine a priori the probability with which the experiment can end with such a result.

Within discrete experiments, we distinguish those, which are performed in phases. They are called multi-phase experiments. Such tosses are n-fold coin tosses. Within multi-phase experiments, we distinguish those in which the number of the phases is random. They are called random experiments with random number of phases. Such a random experiment with a random number of phases is for example when a dice is being thrown until number six is thrown.

Definition 27 If Ω_δ is a set of results of discrete random experiment δ, and p_δ is a function, which assigns to each result of set Ω_δ a probability with which experiment δ can end with such a result, then pair $(\Omega_\delta, p_\delta)$ is a discrete probability space which is called a model of stochastic drawing experiment δ.

7.2 Multi-phase experiments and stochastic tree rules

In [?] is made a discrete probability space for as drawing experiment as their stochastic model ([?], p. 41). In case of a multi-phase experiment, construction tool of discrete probability space (Ω, p) as its model is stochastic tree and two rules:
— rule R1: result of an random experiment performed in phases is a progression of results of consecutive phases, such a result is represented by a tree branch;
— rule R2 (reproduction rule): for each branch of a stochastic tree (and at the same time for each result represented by the branch), there is a corresponding product of numbers assigned to consecutive parts of the branch; the product is called branch weigh ([?], p. 42-43);

Using rule R1, set Ω of results of random experiments is determined. The sum of branch weighs of all branches on the stochastic tree is equal to 1. Using rule R2, we determine function p, which is a probability decomposition on set Ω.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

7.3 Stochastic tree and sums of certain series

Let $a_n = \frac{1}{n\cdot(n+1)}$ where $n \in \mathbb{N}_1$. Let us consider series $\sum a_n$ also its sum

$$\sum_{n=1}^{\infty} \frac{1}{n\cdot(n+1)} = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots.$$

In areas of mathematical analysis the sum of this series may be found with the use $\frac{1}{n\cdot(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, then

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n},$$

it means that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (1 - \tfrac{1}{n}) = 1.$$

Definition 28 U_{b*c} denotes a box in which there are b white balls and c black balls. Let us consider following multi-phase random experiment δ^P_{b*c}. The first phase is a random ball draw from box U_{b*c}. If the randomly drawn ball is black then the experiment ends; if the ball is white, the ball is put back to the box together with another additional white ball. From this new box $U_{(b+1)*c}$ another ball is drawn. It is the second phase. If the randomly drawn ball is black then the experiment ends; if the ball is white, the ball is put back to the box together with another additional white ball. From this new box $U_{(b+2)*c}$ another ball is drawn. It is the third phase and depending on the result of the draw and the procedure is performed analogically as described. This procedure is performed until a black ball is drawn out. This experiment δ^P_{b*c} with random numerical phases is called Polya's scheme ([?], p. 116).

The result of experiment δ^P_{b*c} is unambiguously described by phase number in which a black ball is drawn out first time. Let ω_n denote the result: `a black ball is drawn out in the` n`-th phase` ($n \in \mathbb{N}_1$). In this case, it is possible to code result using rule R1. Set of results of the experiment δ_c is an infinite set $\Omega^P_{b*c} = \{\omega_1, \omega_2, \omega_3, \ldots\}$.

Example 29 Sum of series $\sum a_n$, where $a_n = \frac{1}{n\cdot(n+1)}$ may be found by using stochastic method. Let us consider Polya's scheme δ^P_{1*1}. The figure 1. presents start of fragment stochastic tree this scheme and digraph of function p_{1*1} defined by rule R2.

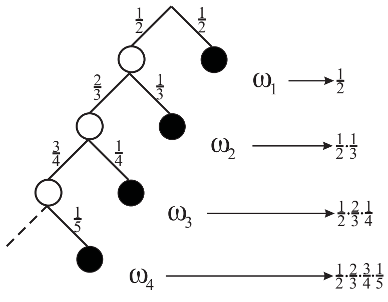

Fig. 1

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

It is important to note that $p_{1*1}(\omega_n) = \frac{1}{n\cdot(n+1)}$ where $n = 1, 2, 3, \dots$. Expression $p_{1*1}(\omega_n)$ is the weigh of the tree branch representing result ω_n. The fact that the sum of all branch tree weighs is equal to 1 results in the following

$$\sum_{\omega_n:\omega_n\in\Omega^P_{1*1}} p_{1*1}(\omega_n) = \sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1.$$

Example 30 Let us consider Polya's scheme δ^P_{1*2} and serie $\sum \frac{1}{n(n+1)(n+2)}$ where $n \in \mathbb{N}_1$ and its sum

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots .$$

Sum of all tree branch weigh of stochastic experiment δ^P_{1*2} is a infinite sum:

$$\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{4} + \frac{1}{3}\cdot\frac{2}{4}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{2}{6} + \cdots + \frac{1}{3}\cdot\frac{2}{4}\cdots\frac{n-4}{n-2}\cdot\frac{n-3}{n-1}\cdot\frac{2}{n} + \cdots .$$

It is sum

$$\frac{2}{3} + \frac{2}{3\cdot 4} + \frac{2\cdot 2}{3\cdot 4\cdot 5} + \frac{2\cdot 2}{4\cdot 5\cdot 6} + \cdots + \frac{2\cdot 2}{(n-2)(n-1)n} + \cdots .$$

The fact that it is the sum of all branch tree weighs, it results in

$$\frac{2}{3} + \frac{2}{3\cdot 4} + \frac{2\cdot 2}{3\cdot 4\cdot 5} + \frac{2\cdot 2}{4\cdot 5\cdot 6} + \cdots + \frac{2\cdot 2}{(n-2)(n-1)n} + \cdots = 1,$$

therefore

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \frac{1}{4}.$$

Example 31 Let us consider series $\sum \frac{1}{(2n-1)(2n+1)}$, where $n \in \mathbb{N}_1$ also Polya's scheme δ^P_{2*2}. It is necessary to say that if p_{2*2} is a function defined on set $\Omega^P_{2*2} = \{\omega_1, \omega_2, \omega_3, \dots\}$ using rule R2, to $p_{2*2}(\omega_n) = \frac{1}{(2n-1)(2n+1)}$ where $n \in \mathbb{N}_1$. It is the branch weight representing the result ω_n in the stochastic tree of Polya's scheme δ^P_{2*2}.

The sum of all branches of the stochastic tree of the experiment δ^P_{2*2} is equal to:

$$\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{3}{5}\cdot\frac{2}{7} + \frac{1}{3}\cdot\frac{3}{5}\cdot\frac{5}{7}\cdot\frac{2}{9} + \cdots + \frac{1}{3}\cdot\frac{3}{5}\cdot\frac{5}{7}\cdots\frac{2n-3}{2n-1}\cdot\frac{2}{2n+1} + \cdots .$$

It is the series

$$\frac{2}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} + \frac{2}{7\cdot 9} + \cdots + \frac{2}{(2n-1)(2n+1)} + \cdots .$$

Considering that the sum of weigh equals 1, we end up with equation

$$\frac{2}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} + \frac{2}{7\cdot 9} + \cdots + \frac{2}{(2n-1)(2n+1)} + \cdots = 1,$$

therefore

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}.$$

In the work, we introduced a method used for determination of sum of some number series using stochastic trees of Polya's scheme: δ^P_{1*1}, δ^P_{1*2} and δ^P_{2*2}.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

Literatura

[1] Batanero C., Henry M., Parzysz, The nature of chance and probability, In G. Jones (Ed.), Exploring probability in school: challenges for teaching and learning (pp. 15-37), Springer, New York 2004.

[2] Batanero C., Godino J. D., Roa R., Training teachers to teach probability, Journal of Statistics Education Vol. 12, on line: www.amstat.org/publications/ise/.

[3] Engel A., Wahrscheinlichkeitsrechnung und Statistik, Band I, Ernst Klett Verlag, Stuttgart 1980.

[4] Engel A., Wahrscheinlichkeitsrechnung und Statistik, Band 2, Ernst Klett Verlag, Stuttgart 1978.

[5] Engel A., Stochastik, Ernst Klett Verlag. Stuttgart 1990.

[6] Engel A., Varga T., Walser W., Strategia czy przypadek? Gry kombinatoryczne i probabilistyczne, WSiP, Warszawa 1979.

[7] Fischbein E., The Intuitive Sources of Probabilistic Thinking in Childem, D.Reidel Publishing Company, Dordrecht 1975.

[8] Freudenthal H., Rola intuicji geometrycznych we współczesnej matematyce, Wiadomości Matematyczne DC (1966).

[9] L. J. Guibas, A. M. Odlyzko, String overlaps, pattern matching, and nontransitive games, Journal of Combinatorial Theory, 30 (1981), seria A, 183–208.

[10] Iwiński T., Szeregi nieskończone, WSiP, Warszawa 1974.

[11] Krech I., Tlust´y P., Stochasticke grafy a jejich aplikace, Jihoceska univerzita v Ceskych Budejovic?ch, Ceske Budejovice 2012.

[12] I. Krech, P. Tlusty, Waiting Time for Series of Successes and Failures and Fairness of Random Games, Scientific Issues, Mathematica II, Ružomberok 2008, 151-154.

[13] Krech I., Probability in probability spaces connected with generalised Penney's games, Acta Univ. Purkynianae 42(1999), 71-77.

[14] Krech I., Serie uspechu a neuspechu s racionaln?? pravdepodobnost? uspechu, Department of Mathematics Report Series 8(2000), 17-24.

[15] Krech I., Prawdopodobieństwo w pewnych argumentacjach na lekcji matematyki, Autenticke vyucovanie a vyuzitie medzipredmetovych vztahov vo vyucoban? matematiky. Zborn?k pr?spevkov, Pedagogicka fakulta UMB, Banska Bystrica, 2. konferencie ucitelov matematiky 2000, 57-61.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

[16] Krech I., Waiting for series of colours and properties of some relations in a set of these series, Acta Univ. Purkynianae 72 Studia Mathematica (2001), 124-128.

[17] Krech I., Łańcuchy Markowa w szkolnej matematyce, Matematika v skole dnes a zajtra. Zborn?k pr?spevkov, Katol?cka Univerzita v Ruzomberku, Matematika v skole dnes a zajtra 2001, 112-124.

[18] Krech I., Osobliwe własności modeli probabilistycznych czekania na serie sukcesow i porażek, Ann. Acad. Paed. Cracov. 5 Studia Ad Calculum Probabilistis Eiusque Didacticam Pertinentia 1(2002), 39-55.

[19] Krech I., Awaiting the series of colours - stochastic graph as the means of mathematical treatment and argumentation, Ann. Acad. Paed. Cracov. 16 Studia Mathematica III (2003), 119-124.

[20] Krech I., Engel's reductions of stochastic graph, Proc. of the XIIth Czech-Polish-Slovak Mathematical School, Hlubo˘s June 2th - 4th 2005, 146-151.

[21] Krech I., Penneyove hry a ich zovseobecnenie, Horizons of mathematics, physics and computer sciences 2/2007 Volume 36, Nitra 2007, 1-7.

[22] Krech I., Stochasticky graf jako hrac? platno k nahodne hre a jako prostredek matematicke argumentace, Acta Universitatis Palackianae Olomucensis, Matematika 3, Olomouc 2008, 155-159.

[23] Krech I., Tlusty P., Stochasticke grafy jako nastroj reseni matematickych uloh, Matematika-fyzika-informatika 5 (Rocnik 19, Leden 2010), Olomouc 2010.

[24] Krygowska Z., Zarys dydaktyki matematyki, t.I., WSiP, Warszawa 1979.

[25] Ma, L. P., Knowing and teaching elementary mathematics, Mahwah. NJ: Lawrence Erlbaum 1999.

[26] M. Major, B. Nawolska, Matematyzacja, rachunki, dedukcja i interpretacja w zadaniach stochastycznych, Wydawnictwo Naukowe WSP, Kraków 1999.

[27] Penney W. F., Problem 95: Penny-Ante, Journal of Recreational Mathematics, (1974)

[28] Penney W. F., Problem 95: Penney-Ante, Journal of Recreational Mathematics, 7 (1974).

[29] Płocki A., Stochastyka dla nauczyciela, Wydawnictwo Naukowe NOVUM, Płock 2010.

[30] Płocki A., Stochastyka dla nauczyczyciela, Wydawnictwo Naukowe NOVUM, Płock 2007.

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.

[31] Płocki A., Dydaktyka stochastyki, Wydawnictwo Naukowe NOVUM, Płock 2005.

[32] Płocki A., Gry Penneya i paradoksy stochastyczne, Matematyka 1 (1999).

[33] Shuo-Yen R. L., A Martingale Approach to the Study of Occurrence of Sequence Patterns in Repeated Experiments, The Annals of Probabability, Volume 8, Number 6 (1980), 1171-1176.

[34] Siwek H., Dydaktyka matematyki. Teoria i zastosowania w matematyce szkolnej, WSiP, Warszawa 2005.

[35] Tversky A., Kahneman D., Availability: A heuristic for judging frequency and probability, Cognitive Psychology 5 (1973).

[36] Tversky A., Kahneman D., Judgments under uncertainty: Heuristics and Biases, Science 185 (1974).

[37] Walter H., Heuristische Strategien und Fehlvorstellungen in stochastischen Situationen, Der Mathematikunterricht, Februar (1983).

Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Education and Culture Executive Agency (EACEA). Neither the European Union nor EACEA can be held responsible for them.