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A methodical guide for teacher

Stochastic graphs

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# A methodical guide for teachers of mathematics in secondary school

# Stochastic graphs

*Ireneusz Krech, Pavel Tlustý*

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## 1 Stochastic graphs vs. in-school probability theory teaching process

### 1.1 Probability theory vs. intuition

Mathematical research and discovery is not only a result of one's pure deduction, inductive thinking and analogy-based reasoning but it is also a result of intuitive thinking (see [29]). The formal approach towards mathematics is often opposed to the intuitive approach. Abstractions and schemas are contrasted to "seeing" and "perception" of general, important mathematical constructions and quantitative and space relations. The inspiration and beginning of all discoveries as well as the point that gives certainty in all kinds of reasoning and the author of new ideas, hypotheses or statements is "obviousness", "common sense", that is – intuition.

For a long time Freudenthal used to replace the word "intuition" with a phrase "shaping of mathematical objects" (see [6]). He was doing so because of a wide range of meanings that the word "intuition" has in different languages. Freudenthal also wrote (see [7]) that "intuitions without concepts are empty, and concepts without intuitions are blind".

Stochastic intuitions are the ability of drawing judgments and beliefs of probabilistic character without any conscious inference or even without perceiving the clues which justify that belief or judgement. It is an ability allowing us to estimate properly the probabilistic characteristics (the event's probability, the expected value, distribution or stochastic independence) of a given sample or population on the basis of incomplete data about the sample and without any (conscious) reasoning or analysis, when the estimation is based only on one's experience or knowledge.

The intuitive conclusions are the ones which we consider obvious, we draw them instantly, almost without thinking, without any reasoning, calculations or argumentations on the basis of images, schemes or situation models that we have in memory. Intuitive thinking is thinking about an abstract situation through its specific model (see [22]).

In [33], [34] and [35] we can find the research of psychologists A. Tversky and D. Kahneman which show that people do not have their probabilistic intuitions properly developed. Humans were not provided even with basic probabilistic intuitions through evolution.

Wrong probabilistic intuitions may be mathematically – based. They can be a result of lack of basic probabilistic, stochastic and combinatorial knowledge, but they can also rise from its poor acquisition (a formalized lecture does not eliminate mistakes in intuitive judgments). They can also have psychological background. A formal explanation of the probability theory and statistics rules is not enough to eliminate those "incorrect representations" in the process of probabilistic predicting, which is seen as an important pre-decisive process by psychologists. The psychological research show that in the process of predicting people do not use probabilistic arguments as much as they use some rules, principles and strategies.

Tversky and Kahneman analyzed the basis of incorrect representations (incorrect intuitions) in situations concerning probability estimations. They point out the divergence between a subjective probability (i.e. estimation of probability given by a person as his /her estimation of a chance of a given event to happen) and objective, normative probability resulting from a probabilistic model. They conducted the research as a part of a bigger project concerning problems of teaching mathematics. They studied the strategies used by people of different age and occupation while solving specific stochastic (combinatorial, as a matter of fact) problems.

J. M. Shaughenessy's research shows how vast is the role of personal contact between a person and empiricism (drawing lots, working with statistical data, using the predevelopped data, like the results of chance games, calculating frequencies, confronting the a posteriori judgments with the ones made a priori) in developing correct stochastic intuitions which appear in using heuristic strategies properly. The same research proves that teaching probabilistic theory in too formalized way, apart from statistics, omitting the empirical aspect of probabilistic issues and leaving out some classical paradoxes like problems – stochastic surprises does, not remove incorrect intuitions. Tversky and Kahneman emphasize the fact, that the same mistakes are made by "stochastically naive" students (the ones with no probabilistic experience) and adults – even ones who had graduated from advanced but formalized stochastic courses. They find mistakes of this kind made even by psychologists who have some knowledge of stochastics.

## 1.2 The functional teaching of mathematics

The idea of functional teaching is the basic strategy of didactically correct process of teaching-learning mathematics. It may also be seen as a basic strategy of discovering

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and creating mathematics by students (see [32]). It is a universal method, recommended in teaching different subjects, but in mathematics – because of an abstract and operative character of mathematical notions – it has got a particular meaning. In functional teaching we try to show mathematics from the notional side, not through the algorithms and rules, as it was in the mechanistic approach. The definitions, rules, reasoning or theorems are important, but they come later on, as a summary, a result of different activities, discovering and using algorithms. According to the integral approach, mathematics should grow from reality, everyday situations. In the functional method the objects and phenomena of the students' environment do not have to be the starting point of mathematical issues. Along with real situations we can use the ones artificially created, using special teaching aids as well as purely abstract problems. The care for precision and order, for clarity and understanding of mathematical issues, for the compatibility of school and scientific notions is vital in the functional teaching. The basis of the student's mathematical activity is his awareness of where in the "math construction" he actually is at the moment. The overriding aim of this teaching method is the student gaining operative knowledge not on the basis of chaotic trials of solving schematic problems or too "casual work", but through the student's activities carefully planned by the teacher. Only a well trained teacher, with a good knowledge of methodology can plan the student's work properly and lead the student to create sequent elements of mathematical knowledge, stressing "mathematical activity, working in mathematical world and its connection to reality, creative experience gathered by the student gradually through solving problems open for creativity at his level" (see [22]).

Through the functional teaching the constructive approach is accomplished. The student creates his own knowledge integrated with various materials and tasks, on the way of reach experience gathered in cooperation with the teacher and fellow students. However, it is not about the superficial shaping of mathematical issues leading to the answer to "what is it" question. It is about active study of techniques and methods that allow the student to solve "the how do we construct" problems. We can find the confirmation of this idea in Piaget's Where does education aim in an extended and supported by numerous research form. Piaget claims there that the basic condition of the whole mind shaping process, which is especially important in the matters that lead young learners to science, is using active methods of teaching. They allow the student to spontaneously search for solutions and demand each truth that is to be discovered to be rediscovered by the student and not only passed to him.

## 1.3 Probability versus stochastic games

Probability is present at every stage of teaching math teachers. But they often lack proper tools of introducing probability at school. This situation is eve a bigger challenge for primary and secondary school teachers. A real didactic suggestion is to introduce stochastic issues on the grounds of chance games that are often followed by lots of stochastic paradoxes. Solving different problems connected to those games leads to

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proper understanding of elementary characteristics and acquiring correct intuitions. Thanks to the paradoxes occurring in those games we can set didactic situations leading to didactic reflections both tor students and teachers. Although probability is present on the elementary and secondary stage of education of math teachers, mathematicians often lack specific tools for teaching probability. Even well trained math teachers, having broad knowledge of mathematics, usually need some additional professional training connected to teaching probability. General rules of teaching which are usually effective in other branches of mathematics are not necessarily as effective in teaching probability theory. This situation is even a greater challenge for primary school teachers. Although teachers do not need a very high level of mathematical knowledge, it is necessary for them to understand the basic notions of mathematics they teach at schools thoroughly, including deep understanding of relations and connections among different aspects of that knowledge (see [23]). The additional elements that are important in the professional teachers' knowledge are described in [1]:

a) epistemology: a reflection on meanings of different notions, like different meanings of probability (see [2]);

b) learning: foreseeing problems in the student's learning, mistakes, obstacles and strategies;

c) didactical means and methods: experience in good selection of examples and didactic situations; ability to analyze the textbooks, curricula and other documents critically; ability to adapt the statistics to different levels of education;

d) ability to engage the students in work and make them interested in what they do; taking their beliefs and attitudes into consideration;

e) interactions: ability to create effective communication in the classroom and using rating as a means of instructing students.

Classical paradoxes play a great role in teaching probability. Because of them we can organize some didactical activities for the math teachers. The aim of these activities is to provoke their reflection on the basic probabilistic notions. These activities also help the teachers understand the students' obstacles and difficulties in understanding probability and they allow them to expand their own methodological and didactical base.

Introduction of the stochastic graph into the probability teaching process is to create, develop and shape those correct stochastic intuitions in a proper way. Simultaneously, we build this process by introducing a specific kind of chance experiments and problems generated by them.

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## 1.4 Penney's game and a stochastic graph

There are two possible results of a coin toss. We shall code them in such a way: $o$ — the result will be heads and $r$ -– the result will be tails. We shall call the $r$ result a success and the $o$ result a failure. The result of $k$ coin tosses, which is a $k$-arrangement of $\{o, r\}$ set we shall call a series of successes and failures, in short – a series of $k$ length.

Let $a$ and $b$ be a defined series of successes and failures of $k$ length. Repeating a coin toss as many times as needed to get $k$ trial result make the $a$ or $b$ series is called waiting for the $a$ or $b$ series and marked as $\delta_{a-b}$. Let us connect the events of:
$A$={waiting $\delta_{a-b}$ will finish with the $a$ series},
$B$={waiting $\delta_{a-b}$ will finish with the $b$ series}
with the $\delta_{a-b}$ chance experiment.

Let us mark the $A$ event as $\{\dots a\}$ and its probability as $P(\dots a)$. The $B$ event shall be marked as $\{\dots b\}$ and its probability as $P(\dots b)$.

In a short article [25] Walter Penney discusses repeating a coin toss as many times as needed to get three times heads or a heads-tails-heads series. Let $\delta_{ooo-oro}$ mean the described chance experiment. Penney suggests a lot game for two players. In the game the $\delta_{ooo-oro}$ experiment is conducted (it is not important who tosses the coin). One of the players wins if the experiment ends up with the $ooo$ result, and the other player wins when the experiment ends with the $oro$ result. The game described above we shall call $g_{ooo-oro}$ . The fact that the $ooo$ and $oro$ series are equally possible to happen would suggest that the game is fair. But the probability that the waiting $\delta_{ooo-oro}$ will end up with the $oro$ series is 0,6, while the probability that it will end up with the $ooo$ series is 0,4. Penney finds the probabilities on a way of particular reasoning (see [27], p. 415) and he does not try to hide his being surprised by the fact that the game is not fair. The $oro$ series gives the player a bigger chance to win that the $ooo$ one. This is the interpretation of the results and the calculation on the real-life ground. So the $oro$ series is called better than the $ooo$ one.

The problem of the fairness of chance games in case of waiting for other pairs of series of heads and tails – those are called Penney's games – the issues connected to the paradox characteristics of the success-failure series in waiting for one of them to occur, as well as the problem of time needed for such waiting (meant as a period of time taken by the game, when time is measured with the number of coin tosses executed) are called Penney's problems in mathematical literature. Only in case of some pairs of heads and tails series the Penney's game is fair. Such series are called equally good.

Some of the results of research on the Penney's problems are gathered in [9] monograph and [30], [11], [12], [13], [14], [15], [19] and [20] articles.

A tool for examining the countable probabilistic spaces for waitings for success-failure series is a stochastic graph. Such a waiting for a series of successes and failures is a chance experiment of a random number of stages.

The research on the probabilistic space for the waiting for one of many success-failure series may be brought down to searching for the probabilities of reaching each of the absorbing levels. Waiting for a success-failure series is often interpreted as a

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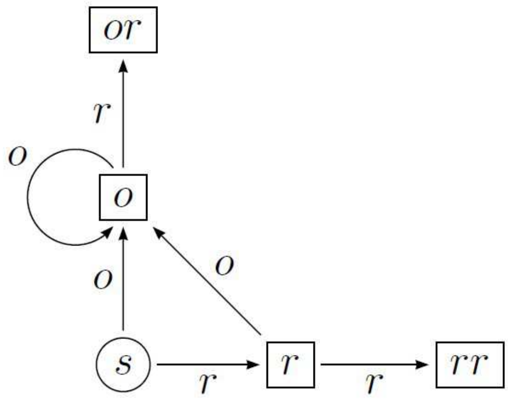


Fig. 1: Stochastic graph - game $g_{rr-or}$

homogeneous Markov's chain with the non-empty set of absorbing stages (see [17]) and it is suggested to use an iconic representation, along with the algebraical one, that is a stochastic graph. Traditionally, such calculations are based on sequences and differential equations. The essence of argumentations based on the stochastic graph is, among others, a reduction of cycles and loops on the graph (we call them reductions of the graph), or transition from a graph with unlimited number of passages to a limited-passage graph (see [16]). It is a development of methods and tools suggested long ago by Arthur Engel in [3], [4] and [5] (see also [18]).

The stages of a homogeneous Markov's chain can be interpreted as points on a plain and called knots. The knot that represents the beginning stage is called starting knot. Each knot representing an absorbing stage is called edge knot. If the probability of getting from a $j$ stage to a $k$ stage in one step is positive, then we connect those knots with the oriented subsection of a line or curve and we mark that subsection $k$. We call that subsection an arc. A graph constructed in such a way is an iconic representation of a Markov's chain.

At the beginning (before conducting the first stage of experiment) we place a pawn in the starting knot of the graph. If a stage ends with the $j$ result we move the pawn along the $j$ arc. The route of the pawn ends when it gets to an edge knot, that is at the rim of the graph (see [21]). Picture 1 shows a stochastic graph being a board of the $g_{or-rr}$ game.

If the pawn gets to the o knot at any stage of the game, it is certain (the probability equals 1) that it will get to the knot (finish) $or$ – that is the player waiting for this series wins. For the pawn getting to the $o$ knot the heads must be the result of first or second toss, so the probability of this event is 0,5+0,25=0,75. The pawn gets to the $rr$ knot only if the first and second toss result with tails, then the other player wins, and this happens with the probability of 0,25.

It is just one example of elementary, simple, but very elegant and making a great impression reasoning based on a stochastic graph. There are lots of such examples can

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be found in the quoted literature.

A natural generalization of discussed problems is replacing a coin toss with any chance experiment having two possible results of non-equal probability (that is a Bernoulli's trial) or a chance experiment having more than two results. Then we can discuss the series of successes and failures or series of colors (flags).

## 1.5 The Waitings for flags **computer program**

An interesting complementation of the discussed issues is a computer program called Waitings for flags (file cnf.exe), which allows us to gather statistical data in a quick and easy way and so formulate different assumptions on their basis.

The program has its limitations:

1) possible number of series: 1 to 4;

2) the number of results in a single experiment: 2 to 12;

3) the probability of each result in a single experiment: measurable, given with maximum accuracy to 12 decimal places.

After inserting the number of results in a single experiment (`the number of results for a` $n$`-trial`) their labels appear. They are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B. The labels can be replaced with other available on the keyboard. We enter the probability of each result next to its label. To check if the probabilities sum up to 1 we shall click the `TOTAL` button. When we click `the classical distribution` button the program will automatically insert the same probability in every window. After setting the result labels in a single experiment we enter the number of color series (flags) and, in appearing windows, the color series coded with the result labels for a single experiment. When all the data is entered we click the `READY` button. The probabilities we look for and estimated time of the experiment (game) will show in a new window. In the right upper corner there is `a Simulation` window. After inserting the number of experiments we wish to simulate and clicking `START` a new window opens. It is a protocol of conducting a required number of experiments. In the new window, in its upper part, there is a number of waitings resulting with specified series and their frequencies. We can simulate up to 1000 experiments.

# 2 Probability in Probability Spaces Connected with Generalised Penney's Games

## 2.1 Discrete probability space and probabiliy in such a space

Let $\Omega$ be an arbitrary at least two-element and at most countable set. A non-negative function $p : \Omega \to R$ which fulfils the condition

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$$\sum_{\omega\in\Omega} p(\omega) = 1,$$

is called a probability distribution on the set $\Omega$.

Let $\mathcal{Z} = 2^{\Omega}$. Define the function $P$ on the set $\mathcal{Z}$ in the following way:

$$P(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ p(\omega), & \text{if } A = \{\omega\}, \\ \sum\limits_{\omega\in A} p(\omega), & \text{if } A \text{ is a set with at least two elements,} \end{cases}$$

It is not difficult to show that the function $P$ fulfils the conditions of the axiomatic definition of probability. Therefore the triple $(\Omega, \mathcal{Z}, P)$ is a probability space. It is called a discrete probability space due to the cardinality of $\Omega$.

The elements of the family $\mathcal{Z}$ are called events and the value of the function $P$ for a set A from the family $\mathcal{Z}$ is called the probability of event $A$. In order to define a discrete probability space $(\Omega, \mathcal{Z}, P)$ it is necessary and sufficient to define a probability distribution $p$ on $\Omega$. For this reason the pair $(\Omega, p)$ may also be called a descrete probability space. In the following considerations the construction of a probability space will be understood as the construction of a pair $(\Omega, p)$ in which $\Omega$ is a set containing at least two elements and at most countable and $p$ is a probability distribution on $\Omega$.

## 2.2 Series of successes and failures, waiting for one of the two series and its probability model

A random experiment with two possible results is called a Bernoulli trial or a trial if the probabilities of these two results are positive. Let one of them be denoted by 1 and called a success while the other is denoted by 0 and called a failure. Let us also denote the probabilities of the success and failure by $u$ and $v$ respectively. Therefore $0 < u < 1$ i $u + v = 1$.

Every result of the experiment in which a particular Bernoulli trial is performed m times (i.e. the result of a Bernoulli scheme of m trials) is called a series of successes and failures. Number $m$ is called the length of the series. A series of successes and failures of length $m$ will be represented as the m-arrangement of the set $\{0, 1\}$.

Let $a$ and $b$ be fixed series of successes and failures of the length $m$. The random experiment of repeating the given trial until the results of the last $m$ trials create series $a$ or series $b$ is called waiting for one of the two series $a$ or $b$ and is denoted by $d_{a-b}$.

Let $\Omega_{a-b}$ be the set of such arrangements of $\{0, 1\}$ with at most $m$ terms in which the last $m$ terms create the series $a$ or $b$, while neither the series $a$ nor the series $b$ is created by any previous subsequence of consecutive $m$ terms. The set $\Omega_{a-b}$ consists of all results of the random experiment $d_{a-b}$. For $\omega \in \Omega_{a-b}$ let $j(\omega)$ stand for the number of the terms of the sequence $\omega$ which are equal to 1. Let the symbol $|\omega|$ denote the length of the sequence $\omega$, i.e. the number of its terms. Define the function $p_{a-b}$ on $\Omega_{a-b}$ by the following formula:

$$p_{a-b}(\omega) = u^{j(\omega)} \cdot v^{|\omega|-j(\omega)} \text{ dla } \omega \in \Omega_{a-b}.$$

The function $p_{a-b}$ is a probability distribution on the set $\Omega_{a-b}$, so the pair $(\Omega_{a-b}, p_{a-b})$ is a probability space. It is called the probability model of the random experiment $d_{a-b}$. The set $\Omega_{a-b}$ is not finite but it is countable, the pair $(\Omega_{a-b}, p_{a-b})$ is an infinite (countable) probability space.

## 2.3 Some generalisation of Penney's game onto a series of successes and failures

Let $a$ i $b$ be fixed series of successes and failures of length $m$. Two players $G_a$ and $G_b$ take part in the game. A particular Bernoulli trial is repeated until the results of last m trials create the series $a$ - in which case the player $G_a$ wins - or the series $b$, which means that the player $G_b$ is a winner. Let us denote the game described above by $g_{a-b}$. It is a generalisation of the game suggested in 1969 by Walter Penney for $u = \frac{1}{2}$ (see W. Penney, Problem 95: Penney-Ante, Journal of Recreational Mathematics 7-1974, p. 321).

In this game the random experiment $d_{a-b}$ is performed, modelled by the probability space $(\Omega_{a-b}, p_{a-b})$. Let $\{a \prec b\}$ denote the event $\{$the series $a$ appears before the series $b\}$ and let $P(a \prec b)$ stand for its probability.

## 2.4 Stochastic graph and probability space induced by it

While repeating the trials it is necessary to continuously control the result of the last $m$ trials in order to decide whether the game is over and who is the winner. This procedure may be rationalised by interpreting the course of the experiment $d_{a-b}$ as wandering of a pawn on a stochastic graph. This interpretation refers to the idea of simulation of the course of homogeneous Markov chains presented by Arthur Engel in [?].

Waiting for one of the series of successes and failures is a homogeneous Markov chain. Let us consider the stochastic graph of this Markov chain. Let $\Omega^*$ be the set of all paths on this graph. To each path let us assign the product of numbers related to the consecutive edges of this path. This product is called the weight of the path. The function which to each path assigns its weight will be denoted by $p^*$. The function $p^*$ is a probability distribution on $\Omega^*$, so the pair $(\Omega^*, p^*)$ is a probability space. It is called the space induced by the stochastic graph. All the subsequent calculations and reasonings are conducted in such a probability space induced by a stochastic graph.

If $(\Omega_{a-b}, p_{a-b})$ is the probability model of the random experiment $d_{a-b}$ defined above and $(\Omega^*, p^*)$ is a probability space induced by the stochastic graph of the random experiment $d_{a-b}$, then both spaces are isomorphic.

## 2.5 Argumentation in countable probability spaces based on a stochastic graph

Let us consider the game $g_{01-00}$. In this game the random experiment $d_{01-00}$ is performed.

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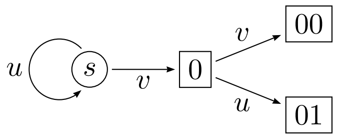
Let the pair $(\Omega_{01-00}, p_{01-00})$ be the probability model for this experiment. In the probability space $(\Omega_{01-00}, p_{01-00})$ the probability $P(01 \prec 00)$ is equal to the sum of a number series. The following paragraphs contain an alternative method of calculating this sum. Figure 6.0.1 presents the stochastic graph of the experiment $d_{01-00}$. At the same time it is the game-board for game $g_{01-00}$).

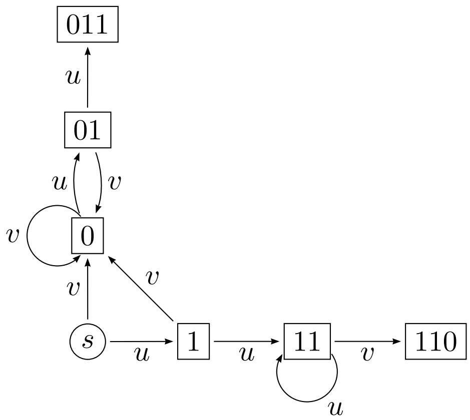



The stochastic graph of the game $g_{01-00}$

From the symmetry of the graph it follows that $P(01 \prec 00) = u$ and $P(00 \prec 01) = v$. Both players wait for the failure to appear. It means that the trial performed directly after the first failure appears settles the question of who wins the game.
If $0_{(n-1)}1$ denotes the series consisting of $(n-1)$ consecutive zeros with 1 as the last term and if $0_n$ denotes the series consisting of zeros only, then generalising the above reasoning we obtain

$$P(0_{(n-1)}1 \prec 0_n 0) = u \text{ oraz } P(0_n 0 \prec 0_{(n-1)}1) = v.$$

Let us now consider the game $g_{011-110}$. Let $(\Omega_{011-110}, p_{011-110})$ be the probability model of the random experiment $d_{011-110}$ conducted in this game.




The stochastic graph of the game $g_{011-110}$

Figure 6.0.2 presents the stochastic graph of the experiment $d_{011-110}$. Let us notice that under the interpretation of the course of the experiment $d_{011-110}$ as the wandering of a pawn on the graph presented in fig.6.0.2 whenever the pawn arrives at the node $\boxed{0}$, with probability equal to 1 the experiment will (sooner or later) end with the series 011 . If the pawn arrives at the node $\boxed{11}$, then with probability equal to 1 the experiment will (sooner or later) end with the series 110. The probability of arriving at the node $\boxed{0}$ is equal to $v + uv$, so $P(011 \prec 110) = v + uv$. Similarly, $P(110 \prec 011) = v^2$.

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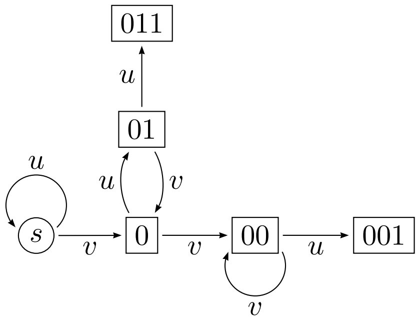
The above reasoning employs some reductions of a graph (see [?], p. 299-302), by which we have passed from the countable probability space $(\Omega_{011-110}, p_{011-110})$, in which $P(011 \prec 110)$ and $P(110 \prec 011)$ are calculated, to a finite space.

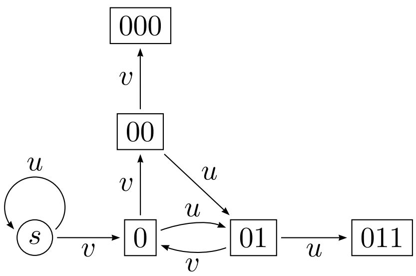
Let us consider the game $g_{001-011}$. Let $(\Omega_{001-011}, p_{001-011})$ be the probability model of the random experiment $d_{001-011}$ conducted in this game. Figure ?? presents the stochastic graph of the random experiment $d_{001-011}$.




The stochastic graph of the game $g_{001-011}$

The node $\boxed{0}$ may be treated as the start node (the probability of arriving at any of the finishes from the start is equal to the probability of arriving at these finishes from the node $\boxed{0}$). The series 001 will appear before the series 011, if the experiment comes to the stage $\boxed{00}$. Let us denote $P(001 \prec 011) = x$. The graph implies that $x = v + uvx$, so $x = \frac{v}{u^2-u+1}$.

Let us consider the game $g_{000-011}$. Let $(\Omega_{000-011}, p_{000-011})$ be the probability model of the random experiment $d_{000-011}$ conducted in this game. Figure ?? presents the stochastic graph of the random experiment $d_{000-011}$.




The stochastic graph of the game $g_{000-011}$

Let $x = P(011 \prec 000)$. In the space induced by the graph $x$ is the probability of arriving at node the $\boxed{011}$ and therefore is the sum of the weights of all paths ending with the node $\boxed{011}$. Referring to the graph presented in figure ?? we obtain $x = u^2 + vu^2 + (uv + uv^2)x$, which implies that $x = \frac{u^2(2-u)}{-u^3+3u^2-2u+1}$.

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The above considerations illustrate how studying countable probability spaces may be reduced to studying finite spaces by applying Engel's graph to interpretation of the course of waiting for one of the two series of successes and failures.

Probabilities of events in the considered probability spaces are for the most part the sums of number series. Finding these probabilities with methods presented in this paper gives at the same time probabilistic methods of calculating sums of some number series.

The present paper presents unknown issues connected with generalisation of Walter Penney's games. Considerations referring to the fairness of the game $g_{a-b}$ with fixed series of successes and failures leads to studying the properties of the function $f(u) = P(a \prec b)$ on the interval $(0, 1)$. The issues presented in the paper illustrate the principle of integration in mathematics teaching.

## 3 On some nontransitive relation

Every result of $k$-tuple toss-up is coded by an arrangement of $k$ out of the set $\{H, T\}$. The element number $j$ of this arrangement is the code of the $j$-th toss-up result.

Definition 1 Every result of $k$-tuple toss-up, where $k \in \mathbb{N}_1$, and therefore every arrangement of $k$ out of the set $\{H, T\}$ is called the series of heads and tails. Number $k$ will be called the length of series. ▼

Definition 2 Let $\alpha, \beta$ be established series of heads and tails of length $k$ and let $\alpha \neq \beta$. We are repeating a toss-up so long, until:
– either the results of $k$ closing toss-ups create the series $\alpha$,
– or the results of $k$ closing toss-ups create the series $\beta$.
Such a random experiment is called awaiting for one of series $\alpha, \beta$ and is denoted by $\delta_{\alpha-\beta}$. ▼

Let $\alpha, \beta$ be series of heads and tails of the length $k$ that is an arrangement of $k$ out of the set $\{H, T\}$. The awaiting $\delta_{\alpha-\beta}$ is a random experiment of random number of stages. Every its result is at least an arrangement of $k$ out of the set $\{H, T\}$ such that a subsequence of $k$ consecutive closing elements of this arrangement is either series $\alpha$ or series $\beta$ and none subsequence of $k$ consecutive preceding elements of


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this arrangement forms any of these series. Let $\Omega_{\alpha-\beta}$ denotes the set of results of the awaiting $\delta_{\alpha-\beta}$. If $\omega \in \Omega_{\alpha-\beta}$ and $\omega$ is $n$-elements sequence, then $\omega$ is a special result of $n$-tuple toss-up. The probability of this result is therefore equal to $\frac{1}{2^n}$ or $\left(\frac{1}{2}\right)^n$ (compare [?], p. 34). The function $p_{\alpha-\beta} : \Omega_{\alpha-\beta} \longrightarrow \mathbb{R}$ is defined by the formula

$$p_{\alpha-\beta}(\omega) = \left(\frac{1}{2}\right)^{|\omega|} \text{ for } \omega \in \Omega_{\alpha-\beta},$$

where $|\omega|$ denotes the number of elements in the sequence $\omega$, it assigns the probability to any result of the experiment $\delta_{\alpha-\beta}$ and at the same time it is a probability distribution on the set $\Omega_{\alpha-\beta}$. The pair $(\Omega_{\alpha-\beta}, p_{\alpha-\beta})$ is a probability space. This is a model of awaiting for one of the series $\alpha, \beta$ (compare [?], p. 41).

We will connect two opposite events with the experiment $\delta_{\alpha-\beta}$:

$$A = \{ \text{ awaiting } \delta_{\alpha-\beta} \text{ will close with getting series } \alpha \ \},$$
$$B = \{ \text{ awaiting } \delta_{\alpha-\beta} \text{ will close with getting series } \beta \ \},$$

which will be denoted as following: $A = \{\alpha \prec \beta\}$ and $B = \{\beta \prec \alpha\}$. The probabilities of events $\{\alpha \prec \beta\}$ and $\{\beta \prec \alpha\}$ we denote appropriately by $P(\alpha \prec \beta)$ and $P(\beta \prec \alpha)$.

Definition 3 If in the probability space $(\Omega_{\alpha-\beta}, p_{\alpha-\beta})$ there is

$$P(\alpha \prec \beta) > P(\beta \prec \alpha),$$

then series $\alpha$ is called better than series $\beta$ and is denoted as $\alpha \gg \beta$. ▼

Definition 4 The end of the length $j$ of the "heads and tails series" $\alpha$ is called the subsequence of $j$ consecutive closing elements of the sequence $\alpha$. The beginning of the length $j$ of the "heads and tail series" $\alpha$ is called the subsequence of $j$ consecutive beginning elements of the sequence $\alpha$. ▼

To state which of the "heads and tails series" is better, one can apply among others Conway algorithm.

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Theorem 5 (Conway algorithm) Let $\alpha, \beta$ be the series of head and tails of the length $k$. Let us assume that

$$\alpha \diamond \beta = \sum_{j=1}^{k} 2^{j-1}$$

for any $j$ meeting such a condition that the end of the length $j$ belonging to the series $\alpha$ is equal to the beginning of the length $j$ belonging to the series $\beta$. Then

$$\frac{P(\alpha \prec \beta)}{P(\beta \prec \alpha)} = \frac{\beta \diamond \beta - \beta \diamond \alpha}{\alpha \diamond \alpha - \alpha \diamond \beta}. \quad \blacksquare$$

This algorithm was given by John H. Conway, and its evidence was published in 1981 by Leo J. Guibas and Andrew M. Odlyzko (compare [8], p. 183-190).

Let $\alpha = THHTTH$, $\beta = HTTHTH$. If we want to calculate the sum $\alpha \diamond \beta$ we have to compare the $\alpha$-series ends of the length $j \in \{1, 2, 3, 4, 5, 6\}$ with the beginnings of $\beta$-series of the same length. Here we have

| | | | | | |
|---|---|---|---|---|---|
| THHTTH | THHTTH | THHTTH | THHTTH | THHTTH | THHTTH |
| HTTHTH | HTTHTH | HTTHTH | HTTHTH | HTTHTH | HTTHTH |
| $2^0$ | 0 | 0 | $2^3$ | 0 | 0 |

We receive then

$$\alpha \diamond \beta = 2^0 + 2^3 = 9,$$

because only for $j \in \{1, 4\}$ the end of $\alpha$-series of the length $j$ is equal to the beginning of $\beta$-series of the length $j$. By analogy we calculate the remaining sums

$$\alpha \diamond \alpha = 2^5 + 2^1 = 34, \quad \beta \diamond \alpha = 2^1 = 2, \quad \beta \diamond \beta = 2^5 + 2^0 = 33.$$

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We receive then
$$\frac{P(\alpha \prec \beta)}{P(\beta \prec \alpha)} = \frac{33-2}{34-9} = \frac{31}{25}.$$

Because the events $\{\alpha \prec \beta\}$ and $\{\beta \prec \alpha\}$ are opposite, then
$$P(\alpha \prec \beta) = \frac{31}{56} \quad \text{and} \quad P(\beta \prec \alpha) = \frac{25}{56}. \quad \square$$

It was shown among others in the work [?] that - paradoxically - the relation $\gg$ in the set $\{H,T\}^3$ is not a transitive relation. Because we have
$$THH \gg HHT, \quad HHT \gg HTT \quad \text{and} \quad HTT \gg THH.$$

Farther we will demonstrate that the relation $\gg$ is transitive in none of the sets $\{H,T\}^k$, $k \geqslant 3$.

Definition 6 (ANTI-OPTIMAL SERIES) Let $\alpha$ be an established series of heads and tails an let $|\alpha| = k$, $k \geqslant 3$. Let us consider the set
$$\mathcal{A}_\alpha = \{\beta : |\beta| = k \wedge \alpha \neq \beta\}.$$

The series $\hat{\alpha}$, meeting the condition
$$\forall_{\beta \in \mathcal{A}_\alpha} P(\hat{\alpha} \prec \alpha) \geqslant P(\beta \prec \alpha),$$

is called the anti-optimal series for the series $\alpha$. ▼

Let us assume that $\alpha = a_1 a_2 \ldots a_k \in \{H,T\}^k$, $k \geqslant 3$. Let us accept the following denotations
$$\alpha_H = Ha_1a_2 \ldots a_{k-1} \quad \text{and} \quad \alpha_T = Ta_1a_2 \ldots a_{k-1}.$$

Theorem 7 If $\alpha$ is the series of heads and tails, then
$$\hat{\alpha} = \alpha_H \quad \text{or} \quad \hat{\alpha} = \alpha_T. \quad \blacksquare$$

The proof of this theorem was given by L. Guibas and A. M. Odlyzko (compare [8], p. 183-208).

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Theorem 8 For every series of heads and tails it is

$$P(\alpha_H \prec \alpha) \neq P(\alpha_T \prec \alpha). \quad \blacksquare$$

It results from the above statements that for any series $\alpha$ there exist exactly one anti-optimal series. This is one of the series $\alpha_H$ or $\alpha_T$. Finally we receive

Theorem 9 For every series of heads and tails it is

$$P(\alpha \prec \hat{\alpha}) < P(\hat{\alpha} \prec \alpha),$$

then the series anti-optimal to series $\alpha$ is the series better than series $\alpha$. $\blacksquare$

The direct conclusion from the Theorem 4 is

Theorem 10 It does not exist such $k \geqslant 3$ that the relation $\gg$ defined in the set $\{H, T\}^k$ is a transitive relation. $\blacksquare$

Proof. Let $\alpha \in \{H, T\}^k$, where $k \geqslant 3$, be any established series of heads and tails. Let us accept the following convention: $\hat{\alpha} = \alpha_1$, $\hat{\alpha_1} = \alpha_2$, $\hat{\alpha_2} = \alpha_3$, etc. Of course it is

$$\alpha_2 \gg \alpha_1 \quad \text{and} \quad \alpha_1 \gg \alpha.$$

If $\sim (\alpha_2 \gg \alpha)$, then the relation $\gg$ is not transitive. But if $\alpha_2 \gg \alpha$, then we consider series $\alpha_3$ and in this case we have

$$\alpha_3 \gg \alpha_2, \qquad \alpha_2 \gg \alpha_1 \quad \text{oraz} \quad \alpha_1 \gg \alpha.$$

If $\sim (\alpha_3 \gg \alpha \vee \alpha_3 \gg \alpha_1)$, then the relation $\gg$ is not transitive. If $\alpha_3 \gg \alpha$ i $\alpha_3 \gg \alpha_1$, then we consider series $\alpha_4$ etc. The set $\{H, T\}^k$ is finite, so the above procedure will lead us after all to the conclusion that the relation $\gg$ is not transitive. $\triangledown$

The relation $\gg$ appears in the context of random experiment carried out in a game suggested in 1974 by Walter Penney (see [?]). Two players: $G_A$ and $G_B$ are participating in the game. At the beginning of the game,

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the players choose series for themselves from the set $\{H,T\}^k$, where $k \geqslant 3$ is an earlier established series' length. Player $G_A$ is choosing his series $\alpha$ as first. Next, the player $G_B$, who is knowing series chosen by the player $G_A$, chooses his series $\beta \in \{H,T\}^k \setminus \alpha$. Then the experiment $\delta_{\alpha-\beta}$ is carried out and if an event $\{\alpha \prec \beta\}$ will happen, then the player $G_A$ is winning, but if an event $\{\beta \prec \alpha\}$ will happen, then the player $G_B$ is winning. It's obvious that a bigger chance to win has this player, whose series is better. It is a paradox, that the player choosing his series as a second player, has - after appropriate choice of series - a greater chance to win in the game. Therefore a priority right at choice of series is not a privilege. Since the relation $\gg$ is not transitive, then - paradoxically - from the fact that the player $G_A$ has greater chance to win than the player $G_B$, and the player $G_B$ has greater chance to win than the player $G_C$ it does not result, that in the game with participation of two players $G_A$ and $G_C$, the player $G_A$ would have a greater chance to win (regardless of what lengths of series do the players choose).

## 4 Better vs. longer series of heads and tails

Definition 11 Let $k \in \mathbb{N}$ and $k \geqslant 1$. Each result of the $k$-fold variation of the $\{H,T\}$ set, which is each result of the $k$-fold coin toss, we shall call a series of heads and tails. We shall mark its length as $|\alpha|$.

Definition 12 Let $\alpha$ and $\beta$ be series of heads and tails. We can say that the seria $\alpha$ series is not included in the $\beta$ series if it is not the subsequence of the successive elements of the $\beta$ series.

Definition 13 Let $\alpha$ and $\beta$ be series of heads and tails. Let the $\alpha$ series be $k$ long and the $\beta$ series be $l$ long. Let us also assume that the $\alpha$ series is not included in the $\beta$ one. We repeat a coin toss so long that we get $k$ last results forming the $\alpha$ series or $l$ last results forming the $\beta$ series. We call this experiment waiting for one of the two stated series of results and mark it as $\delta_{\alpha-\beta}$ (see [??], 406-415).

Let us consider a game of two players, $G_\alpha$ and $G_\beta$. In the game they conduct the $\delta_{\alpha-\beta}$ experiment. If the waiting finishes with the $\alpha$ series -

the player $G_\alpha$ wins, and if it finishes with the $\beta$ series - the $G_\beta$ player wins. We shall call this game the Penney's game[1] and mark it as $g_{\alpha-\beta}$.

Let us consider the waiting of $\delta_{\alpha-\beta}$. The $\omega$ sequence having its elements from the set of $\{H, T\}$ is a result of the $\delta_{\alpha-\beta}$ experiment if it fulfills the following conditions:

- the subsequence of $k$ last results forms the $\alpha$ series or the subsequence of $l$ last results forms the $\beta$ series, and
- no subsequence of $k$ or $l$ successive results forms the $\alpha$ or $\beta$ series.

We mark the set of all such sequences (results of the $\delta_{\alpha-\beta}$ experiment) as $\Omega_{\alpha-\beta}$.

If the $\omega$ result of the $\delta_{\alpha-\beta}$ experiment is an $n$-element sequence, it is a specific result of an $n$-fold coin toss. Its probability equals $\left(\frac{1}{2}\right)^n$.
Let $p_{\alpha-\beta}$ be a function of

$$p_{\alpha-\beta}(\omega) = \left(\frac{1}{2}\right)^{|\omega|} \text{ for } \omega \in \Omega_{\alpha-\beta},$$

and $|\omega|$ be the $\omega$ sequence length (number of elements). This function is the distribution of probability in the $\Omega_{\alpha-\beta}$ set, and the $(\Omega_{\alpha-\beta}, p_{\alpha-\beta})$ pair is a probabilistic model of the $\delta_{\alpha-\beta}$ waiting.

Let us state two opposite events in the space of $(\Omega_{\alpha-\beta}, p_{\alpha-\beta})$:

$$A = \{\text{the } \delta_{\alpha-\beta} \text{ waiting gives the } \alpha \text{ series at the end}\},$$

$$B = \{\text{the } \delta_{\alpha-\beta} \text{ waiting gives the } \beta \text{ series at the end}\}.$$

Definition 14 If $P(A) = P(B)$, we call the $\alpha$ and $\beta$ series equally good and mark them as $\alpha \approx \beta$.

Definition 15 If $P(A) > P(B)$, we call the $\alpha$ series better than the $\beta$ series and mark them as $\alpha \gg \beta$.

In the game of $g_{\alpha-\beta}$ we conduct the $\delta_{\alpha-\beta}$ experiment. If the $A$ event occurs, the $G_\alpha$ player wins. If the experiment ends with the $B$ event, the game winner is the $G_\beta$ player. Stating the probability of the $A$ and $B$

[1] Proposed by Walter Penney, see [??];

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events we can also determine the fairness of the Penney's game. If the $\alpha$ and $\beta$ series are equally good, the players have equal chance to win. The $g_{\alpha-\beta}$ game is fair. If one of the series is better than the other, the players' chances to win are not equal and the game is not fair.

Let $\delta_{\alpha-\beta}$ be waiting for one of two series of heads and tails and $k$ and $l$ be lengths of $\alpha$ and $\beta$ series. Let $m \in \{1, 2, 3, \ldots, min\{k, l\}\}$, $\alpha^{(m)}$, $\beta^{(m)}$ mean respectively sequences of $m$ first elements of $\alpha$ and $\beta$ series and $\alpha_{(m)}$, $\beta_{(m)}$ mean respectively $m$ last elements of the $\alpha$ and $\beta$ series. Let us define the sets

$$A_\alpha = \{m : \alpha_{(m)} = \alpha^{(m)}\}, \qquad A_\beta = \{m : \alpha_{(m)} = \beta^{(m)}\},$$

$$B_\beta = \{m : \beta_{(m)} = \beta^{(m)}\}, \qquad B_\alpha = \{m : \beta_{(m)} = \alpha^{(m)}\},$$

and the following sums

$$\alpha : \alpha = \sum_{j \in A_\alpha} 2^j, \qquad \alpha : \beta = \sum_{j \in A_\beta} 2^j,$$
$$\beta : \beta = \sum_{j \in B_\beta} 2^j, \qquad \beta : \alpha = \sum_{j \in B_\alpha} 2^j.$$

Theorem 16 In the probabilistic space of $\delta_{\alpha-\beta}$ the equation

$$\frac{P(B)}{P(A)} = \frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha},$$

called the Conway's equation is true[2].

Remark 17 From the preceding equation we can tell that if

$$\mu := \frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha},$$

than

$$\mu > 1 \Leftrightarrow \beta \gg \alpha,$$
$$\mu = 1 \Leftrightarrow \alpha \approx \beta,$$
$$\mu < 1 \Leftrightarrow \alpha \gg \beta.$$

[2] Discovered by John Horton Conway; the proof of its correctness is shown in [??];

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Let $\alpha = HTHTHT$ and $\beta = HHTHTH$. Let us notice that $\alpha_{(1)} = T \neq H = \alpha^{(1)}$, so $1 \notin A_\alpha$. Analogously

$$\left.\begin{matrix}\mathrm{HT}HTHT\\ HTHT\mathrm{HT}\end{matrix}\right\} \Rightarrow\ 2 \in A_\alpha, \qquad \left.\begin{matrix}\mathrm{HTH}THT\\ HTH\mathrm{THT}\end{matrix}\right\} \Rightarrow\ 3 \notin A_\alpha,$$

$$\left.\begin{matrix}\mathrm{HTHT}HT\\ HT\mathrm{HTHT}\end{matrix}\right\} \Rightarrow\ 4 \in A_\alpha, \qquad \left.\begin{matrix}\mathrm{HTHTH}T\\ H\mathrm{THTHT}\end{matrix}\right\} \Rightarrow\ 5 \notin A_\alpha,$$

$$\left.\begin{matrix}\mathrm{HTHTHT}\\ \mathrm{HTHTHT}\end{matrix}\right\} \Rightarrow\ 6 \in A_\alpha.$$

Therefore

$$A_\alpha = \{2, 4, 6\},$$

so

$$\alpha : \alpha = 2^2 + 2^4 + 2^6 = 84.$$

In the same way we come to the following

$$\alpha : \beta = 0, \qquad \beta : \beta = 66, \qquad \beta : \alpha = 42,$$

so

$$\frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha} = \frac{84 - 0}{66 - 42} = \frac{21}{6} > 1.$$

Therefore $HHTHTH \gg HTHTHT$, and this means that the $g_{HTHTHT-HHTHTH}$ is not a fair one.

Let $\delta_{\alpha-\beta}$ be waiting for one of the $\alpha$ or $\beta$ series. Let us assume that $|\alpha| > |\beta|$. Intuitionally we can presume that the $\beta$ series, being shorter than the $\alpha$ series, is a better one.

Let us consider two series: $\alpha = HHTT...TT$ and $\beta = TT...TT$. The series are such that $|\alpha| = |\beta| + 1 = k + 1$, where $k \geqslant 2$. In this case

$$\alpha : \alpha = 2^{k+1}, \qquad \alpha : \beta = \sum_{j=1}^{k-1} 2^j,$$

$$\beta : \beta = \sum_{j=1}^{k} 2^j, \qquad \beta : \alpha = 0.$$

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From the Conway's equation we know that

$$\frac{P(A)}{P(B)} = \frac{\beta:\beta-\beta:\alpha}{\alpha:\alpha-\alpha:\beta} = \frac{\sum_{j=1}^{k} 2^j}{2^{k+1}-\sum_{j=1}^{k-1} 2^j}.$$

Let us notice that $\sum_{j=1}^{n} 2^j$ is a sum of $n$ first elements of the geometrical sequence which has the first element of 2 and the quotient of 2, so

$$\sum_{j=1}^{n} 2^j = 2\frac{1-2^n}{1-2} = 2^{n+1}-2. \tag{4.0.1}$$

Therefore

$$\frac{\sum_{j=1}^{k} 2^j}{2^{k+1}-\sum_{j=1}^{k-1} 2^j} = \frac{2\cdot 2^k-2}{2\cdot 2^k-2^k+2} = \frac{1-\frac{1}{2^k}}{\frac{1}{2}+\frac{1}{2^k}} > \frac{1}{\frac{1}{2}},$$

and

$$\frac{P(A)}{P(B)} > 2,$$

so $\alpha \gg \beta$ even if the $\alpha$ series is longer than the $\beta$ one.

If we narrow our consideration to pairs of series that differ by more than one element in length, we can easily see that the shorter series is a better one.

Theorem 18 Let $\delta_{\alpha-\beta}$ be waiting for one of the $\alpha$ or $\beta$ series of heads and tails which lengths fulfill the condition $|\alpha| \geqslant |\beta|+2$. Then the $\beta$ series is better than the $\alpha$ series.

Proof. Let the $\alpha$ and $\beta$ be series of heads and tails and $|\alpha| = k$, $|\beta| = l$. Let $m \geqslant 2$ be such a number that $k = l+m$. As the series cannot include each other, we have

$$\{k\} \subset A_\alpha \subset \{1,2,3,...,k\}, \qquad A_\beta \subset \{1,2,3,...,l-1\},$$

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$$\{k\} \subset B_\beta \subset \{1,2,3,...,l\}, \qquad B_\alpha \subset \{1,2,3,...,l-1\},$$

which lead us to the following approximations:

$$2^k \leqslant \alpha:\alpha \leqslant \sum_{j=1}^{k} 2^j, \qquad 0 \leqslant \alpha:\beta \leqslant \sum_{j=1}^{l-1} 2^j$$

$$2^l \leqslant \beta:\beta \leqslant \sum_{j=1}^{l} 2^j, \qquad 0 \leqslant \alpha:\beta \leqslant \sum_{j=1}^{l-1} 2^j.$$

Then

$$\frac{\beta:\beta-\beta:\alpha}{\alpha:\alpha-\alpha:\beta} \leqslant \frac{\sum_{j=1}^{l} 2^j - 0}{2^k - \sum_{j=1}^{l-1} 2^j}.$$

From (4.0.1) we get

$$\frac{\sum_{j=1}^{l} 2^j}{2^k - \sum_{j=1}^{l-1} 2^j} = \frac{2\cdot 2^l - 2}{2^{m+l} - (2^l - 2)} < \frac{2\cdot 2^l}{2^m \cdot 2^l - (2^l-2)} = \frac{2}{2^m - (1-\frac{2}{2^l})} < \frac{2}{4-1},$$

therefore

$$\frac{\beta:\beta-\beta:\alpha}{\alpha:\alpha-\alpha:\beta} < 1.$$

Considering the remark 17 we get $\beta \gg \alpha$.

## 5 Waiting for a series of colours, and the properties of certain relations in a set of these series

Let us consider an urn $U$ containing white, black and green balls. There are three possible results of sampling a ball from this urn:

$b$: `the sampled ball will be white`,
$c$: `the sampled ball will be black`,
$z$: `the sampled ball will be green`.

Let $p(x)$ denotes the probability that the result of the sampling is $x$, $x \in \{b,c,z\}$. Let us assume, that

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$$p(b) = u > 0,\ p(c) = v > 0 \text{ i } p(z) = w = 1 - u - v > 0.$$

The result of a composed m-times sampling of a ball from the urn $U$ with return is an m-arrangement of the set $\{b, c, z\}$ (j-th term is a result of j-th sampling). Every such result we call the series of colours of length $m$.

Let $\alpha_1$ and $\alpha_2$ be the settled series of colours of length $m$. Sampling from the urn $U$ as long as the result of $m$ last samplings will create a series $\alpha_1$ or a series $\alpha_2$ we will call awaiting one of the series $\alpha_1$, $\alpha_2$ and we will denote it by $d_{\alpha_1-\alpha_2}$. Let us connect the following events with the experiment $d_{\alpha_1-\alpha_2}$:
$\{\dots\alpha_1\} = \{$awaiting $d_{\alpha_1-\alpha_2}$ will be finished by obtaining the series $\alpha_1\}$,
$\{\dots\alpha_2\} = \{$awaiting $d_{\alpha_1-\alpha_2}$ will be finished by obtaining the series $\alpha_2\}$.

If $P(\dots\alpha_1) > P(\dots\alpha_2)$, we say that the series $\alpha_1$ is *better* than the series $\alpha_2$ and we denote it by $\alpha_1 \gg \alpha_2$. If $P(\dots\alpha_1) = P(\dots\alpha_2)$, we say that the series $\alpha_1$ and $\alpha_2$ are equally good and we denote it $\alpha_1 \approx \alpha_2$. There is $P(\dots\alpha_1) = 1 - P(\dots\alpha_2)$, therefor $P(\dots\alpha_1)$ is the function of two variables $u$ and $v$. Let us remark, that

$$\alpha_1 \approx \alpha_2 \Longleftrightarrow P(\dots\alpha_1) = \tfrac{1}{2},\ \alpha_1 \gg \alpha_2 \Longleftrightarrow P(\dots\alpha_1) > \tfrac{1}{2}.$$

The following problems are the subject of this paper:
– for which values of the $u$ and $v$ parameters there is $\alpha_1 \approx \alpha_2$;
– for which values there is $\alpha_1 \gg \alpha_2$ and for which there is $\alpha_2 \gg \alpha_1$.

The motivation for these calculations is the special sampling game. A random experiment $d_{\alpha_1-\alpha_2}$ is carried out in the game with two players $G_1$ and $G_2$ and if the event $\{\dots\alpha_j\}$ takes place, the player $G_j$ wins ($j = 1, 2$). The described sampling game is a kind of generalisation of the game proposed by Walter Penny in [26].

The events $\{\dots\alpha_j\}$ and their probabilities are considered here in the model of the experiment $d_{\alpha_1-\alpha_2}$. This is the probability space $(\Omega_{\alpha_1-\alpha_2}, p_{\alpha_1-\alpha_2})$, where:
– $\Omega_{\alpha_1-\alpha_2}$ is a set of arrangements of terms from the set $\{b, c, z\}$, and the subsequence of last $k$ terms is a series $\alpha_1$ or a series $\alpha_2$ and no subsequence $k$ previous elements is not a series $\alpha_1$ nor $\alpha_2$;
– $p_{\alpha_1-\alpha_2}$ is a function defined by the formula:

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$$p_{\alpha_1-\alpha_2}(\omega) = u^{l_b} \cdot v^{l_c} \cdot w^{l_z} \text{ dla } \omega \in \Omega_{\alpha_1-\alpha_2},$$

where $l_x$ denotes the number of terms equal to $x$ in the sequence $\omega$ and $x \in \{b, c, z\}$, and the probability in this space is the function $P$ defined by the formula:

$$P(A) = \sum_{\omega \in A} p_{\alpha_1-\alpha_2}(\omega) \text{ dla } A \subset \Omega_{\alpha_1-\alpha_2}.$$

Generally the probabilities of events in the space of probability $(\Omega_{\alpha_1-\alpha_2}, p_{\alpha_1-\alpha_2})$ are the sums of series. The other, simpler tools of calculation are proposed in this paper.

The experiment $d_{\alpha_1-\alpha_2}$ is the uniform Markow string if it is being analysed from the point of view of the state of the waiting after the subsequent sampling of the ball (see [27]). This is Markow string of two absorbing states. Each of these states has its own stochastic graph (see [3]).

Let us consider the random experiment $d_{bc-bb}$ and the events:
$\{\dots bc\} = \{$the waiting $d_{bc-bb}$ will be finished by obtaining the series $bc\}$,
$\{\dots bb\} = \{$the waiting $d_{bc-bb}$ will be finished by obtaining the series $bb\}$.

The stochastic graph of the random experiment is presented in the Fig. 1.

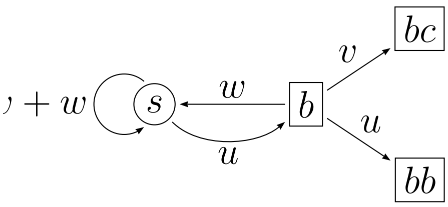


Fig. 1

The pattern of the random experiment $d_{bc-bb}$ may be interpreted as a random walk of a pawn on a stochastic graph like in Fig. 1. At the beginning we place the pawn on the place start and then we move it along the edge with associated number $u$, if the sampling was finished by the white ball, along the edge with associated number $v$, if the sampling was finished by the black ball or along the edge with associated number $w$, if the sampling was finished by the green ball. The experiment is finished

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if the pawn is at the place $\boxed{bc}$ (in this case the event $\{\ldots bc\}$ took place) or if it is at the place $\boxed{bb}$ (in this case the event $\{\ldots bb\}$ took place).

Any such string of edges that the start point of the first one is at start and the end of the last one is $\boxed{bc}$ or $\boxed{bb}$ and at the same time the start point of any edge is the end point of the previous edge, is called the track on the graph. There is a bijection from the set of tracks in the graph onto the set $\Omega_{bc-bb}$ maintaining the probability. Graph in the Fig. 2a is the sub-graph of the graph from the Fig. 1. This sub-graph reduces itself to the graph from Fig. 2b. This reduction is based on the fact that there is the probability $\frac{u}{1-(v+w)}$ i.e. probability equal 1 that the pawn will come from start to the place $\boxed{b}$.

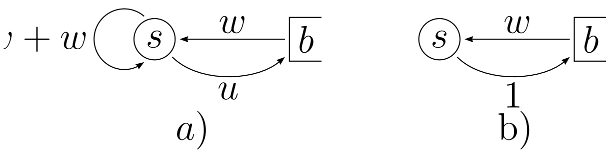


Fig. 2

Therefore the stochastic graph of the random experiment $d_{bc-bb}$ in Fig. 1 reduces itself to the graph presented in Fig. 3.

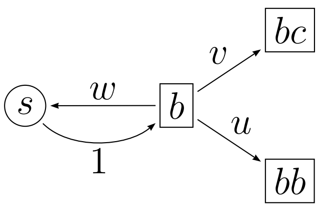


Fig. 3

According to the above interpretation (and taking into account the reduction of the graph) $P(\ldots x)$ is the probability of coming from start to the place $\boxed{x}$ ($x \in \{bc, bb\}$) in random walking across the graph in Fig. 3. Therefore it is:

$$P(\ldots bc) = v + w \cdot v + w^2 \cdot v + w^3 \cdot v + \ldots = v + \sum_{k=1}^{\infty} w^k \cdot v = v + \frac{w \cdot v}{1-w} = \frac{v}{u+v}.$$

Analogously we have:

$$P(\ldots bb) = u + w \cdot u + w^2 \cdot u + w^3 \cdot u + \ldots = u + \sum_{k=1}^{\infty} w^k \cdot u = u + \frac{w \cdot u}{1-w} = \frac{u}{u+v}.$$

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Let $P(\dots bc) = f(u,v)$. Therefore it is $P(\dots bb) = 1 - f(u,v)$. The domain of the function $f$ is a set of the pairs of numbers $(u,v)$ satisfying the following condistion:

$u > 0$, $v > 0$ oraz $u + v < 1$ (here and below $u$ and $v$ are

the rational numbers).

If we interpret the pair of numbers $(u,v)$ as a point in Cartesian co-ordinates, the domain of the function $f$ is a figure bounded by the co-ordinate axes and the line $v = 1 - u$ (In the Fig. 4 this is the triangle $ABC$).

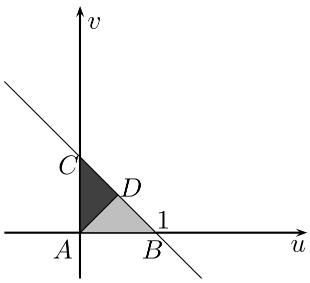


Fig. 4

Let us note, that:

$$P(\dots bc) = P(\dots bb) \iff P(\dots bc) = \tfrac{1}{2} \iff \tfrac{v}{u+v} = \tfrac{u}{u+v},$$

therefore $bc \approx bb \iff [u = v \wedge u, v \in (0, \frac{1}{2})]$.

The set of the points satisfying this condition in the Fig. 4 is the segment $AD$.

Analogously

$$P(\dots bc) > P(\dots bb) \iff P(\dots bc) > \tfrac{1}{2} \iff \tfrac{v}{u+v} > \tfrac{u}{u+v},$$

therefore $bc \gg bb \iff [v \in (0, \frac{1}{2}) \wedge v < u < 1 - v]$.

In the Fig. 4 the set of the points satisfying this condition is the triangle $ABD$.

We have also:

$$P(\dots bc) < P(\dots bb) \iff P(\dots bc) < \tfrac{1}{2} \iff \tfrac{v}{u+v} < \tfrac{u}{u+v},$$

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therefore $bb \gg bc \iff [u \in (0, \frac{1}{2}) \wedge u < v < 1 - u]$.

In the Fig. 4 the set of the points satisfying this condition is the triangle $ADC$.

The subject of this work are peculiar arguments concerning the probability in some countable probabilistic spaces.

## 6 Awaiting the series of colours - stochastic graph as the means of mathematical treatment and argumentation

The construction of a probabilistic space as a model of a certain multistage sampling experiment (see [27], pages 24–25) is a form of mathematical treatment. The argumentations apply to probabilities in this space.

Let us consider a box $U_n$ containing $r$ balls of $n$ different colours: $k_1, k_2, \ldots, k_n$ ($n \leqslant r$). The results of sampling the balls from the box $U_n$ create a set $\Omega = \{k_1, k_2, \ldots, k_n\}$. Let $u_j$ denote the probability of sampling the ball of colour $k_j$ from the box $U_n$ ($j \in \{1, 2, \ldots, n\}$). The function $p$ meeting the following conditions: $p(k_1)=u_1>0$, $p(k_2)=u_2>0$, …, $p(k_n)=u_n>0$ is the probability distribution on the set $\Omega$, and therefore the pair $(\Omega, p)$ is the probabilistic space. This is the probabilistic model of sampling a ball from the box $U_n$. Sampling a ball from the box $U_n$ we call a trial and we denote it by $d_{u_1,\ldots,u_n}$ or by $d_n$ if $u_1 = u_2 = \cdots = u_n = \frac{1}{n}$. The elements of the set $\Omega$ are called colours. The trial is defined unequivocally by the $n$-terms sequence $(u_1, u_2, \ldots, u_n)$ of rational numbers from the interval $(0, 1)$ sum of which is equal to 1, where $u_j = p(k_j)$.

The result of $m$-time recurrence of the trial $d_{u_1,\ldots,u_n}$, i.e. any $m$-terms arrangement of the set $\{k_1, k_2, \ldots, k_n\}$, is called the series of colours or is called a flag and is denoted by $\alpha$. The number $m$ is called the length of colours series and is denoted by $|\alpha|$. If $t < m$, the sub-sequence $t$ of subsequent ending elements of colours series $\alpha$ is called its ending of the length t.

We say that a flag $\alpha_1$ is contained in a flag $\alpha_2$, what is denoted by $\alpha_1 \subset \alpha_2$, if $\alpha_1$ is a sub-sequence of the consecutive terms of the sequence $\alpha_2$. If a flag $\alpha_1$ is not a sub-sequence of the consecutive terms of the sequence $\alpha_2$, we say that the flag $\alpha_1$ is not contained in the flag $\alpha_2$, what

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is denoted by $\alpha_1 \not\subset \alpha_2$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the fixed flags of $|\alpha_j| = n_j$ so that for any $l, j \in \{1, 2, \ldots, k\}$ and $l \neq j$ the following condition is satisfied:

$$\alpha_l \neq \alpha_j \quad \wedge \quad [(|\alpha_l| \neq |\alpha_j|) \Rightarrow (\alpha_l \not\subset \alpha_j \wedge \alpha_j \not\subset \alpha_l)].$$

The repetition of the trial $d_{u_1,\ldots,u_n}$ for such a long time that:
— the results of $n_1$ last trials create the flag $\alpha_1$,
— or the results of $n_2$ last trials create the flag $\alpha_2$,
⋮
— or the results of $n_k$ last trials create the flag $\alpha_k$,
is called the awaiting one of k flags and is denoted by $d^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$ (or by $d^{n}_{\alpha_1,\ldots,\alpha_k}$, if $u_1 = u_2 = \cdots = u_n = \frac{1}{n}$).

Any result of the experiment $d^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$ (as an elementary event) is such a sequence $\omega$, that:
— $\omega \in \{1, 2, \ldots, n\}^m$, where $m \geqslant min\{n_1, n_2, \ldots, n_k\}$ and
— either the ending of the length $n_1$ of the sequence $\omega$ creates the flag $\alpha_1$,
— or the ending of the length $n_2$ of the sequence $\omega$ creates the flag $\alpha_2$,
⋮
— or the ending of the length $n_k$ of the sequence $\omega$ creates the flag $\alpha_k$,
and none other sub-sequence of the previous terms of the sequence $\omega$ creates any flag of $\alpha_1, \alpha_2, \ldots, \alpha_k$.

The set of all possible results of the experiment $d^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$ is denoted by $\Omega^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$.

Let $J_j(\omega)$ denotes the number of terms equal to $k_j$ in the sequence $\omega$, where $j = 1, 2, \ldots, n$. We define the function $p : \Omega^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k} \longrightarrow R$ by the formula:

$$p^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}(\omega) = u_1^{J_1(\omega)} \cdot u_2^{J_2(\omega)} \cdot \cdots \cdot u_n^{J_n(\omega)}.$$

The function $p^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$ is the probability distribution on the set $\Omega^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$. The pair $(\Omega^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}, p^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k})$ is the probabilistic space , that we consider as a probabilistic model of the sampling experiment $d^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$.

Let us associate the below events with the sampling experiment $d^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$:
$A_j = \{$awaiting $d^{u_1,\ldots,u_n}_{\alpha_1,\ldots,\alpha_k}$ will result with the flag $\alpha_j\}$, for $j = 1, 2, \ldots, n$.

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The event $A_j$ we denote by $\{\ldots\alpha_j\}$ and its probability - by $P(\ldots\alpha_j)$.

Let us consider a box $U_3$ containing three balls of colours: white, black and green. There are only three possible results of sampling a ball from this box:
$b$ : the sampled ball will be white,
$c$ : the sampled ball will be black,
$z$ : the sampled ball will be green.

Sampling of a ball from the box $U_3$ is the trial $d_3$. Composed sampling with return of the ball back to the box $U_3$ until black ball will be sampled two times running or until black ball will be sampled after the white one, is the waiting for one of two flags: $\alpha_1 = cc$ and $\alpha_2 = bc$ or in other words: it is the sampling experiment $d^3_{cc-bc}$.

Let us distinguish the following states in the experiment $d^3_{cc-bc}$: the initial state $s$ and the states $c, b, cc, bc$. The procedures described below lead up to the stochastic graph of the sampling experiment $d^3_{cc-bc}$. Every state of the experiment is interpreted as point of a plane and is called a node of the graph. The graph node representing the state $s$ is denoted by start. The graph node representing a state $j$ different from $s$ is denoted by $\boxed{j}$. Let $p_{j\to k}$ denotes the probability that the experiment $d^3_{cc-bc}$ would be in state $k$ after a given sampling, if after the previous sampling it was in state $j$. If $p_{j\to k} > 0$ then we join the node $\boxed{j}$ with the node $\boxed{k}$ by an oriented segment of straight or curved line. This segment is called the edge of the graph and it is denoted by $j \to k$. Close to each graph edge we write down:
— the result of the trial, after which waiting it would be in the state $k$ if after the previous trial it was in the state $j$, if $u_1 = u_2 = \cdots = u_n = \frac{1}{n}$ or

— the number $p_{j\to k}$ if $j, k \in \{1, 2, \ldots, n\}$ exist such that $u_j \neq u_k$.

If $p_{j\to j}>0$ then the edge $j \to j$ is called the loop. If $p_{j\to j} = 1$ then we neglect the loop $j \to j$ and the node $\boxed{j}$ is called the peripheral node. The set of all peripheral nodes is called the periphery of the graph.

Any sequence of edges such that the beginning of the first one is the node start, the end of the last edge is a peripheral node and in case of any two other sequential edges the beginning of it is at the same time

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the end of the next one (excluding the last edge) is called a trace. The weight of the trace is the product of numbers associated to its sequential edges.

Any sequence of edges such that the beginning of the first one is the node $\boxed{j}$, the end of the last edge is a peripheral node and in case of any two other sequential edges the beginning of it is at the same time the end of the next one is called a cycle and we denote it by $c_k$ ($k$ is an index ordering the set of cycles). The number of edges creating the cycle $c_k$ is called the cycle length and is denoted by $r(c_k)$. A loop is considered as a cycle of length 1. The weight of the cycle is the product of numbers associated to its edges. The nodes being the beginnings or endings of the cycle edges are called the inner nodes of the cycle. If $j, k_1, \ldots, k_n$ are the inner nodes of the cycle and $p_{j \to k_1} > 0$, $p_{k_n \to j} > 0$ and $p_{k_l \to k_{l+1}} > 0$ for $l = 1, 2, \ldots, n-1$, so such a cycle we denote by $j \to k_1 \to \ldots \to k_n \to j$. If a sequence of trace edges creates a cycle, we say that this trace contains a cycle. Otherwise we say that the trace does not contain a cycle.

Figure 5 presents the stochastic graph of the sampling experiment $d^3_{cc-bc}$. (see [27] and [3]).

Let us consider the set $\Omega^*_{cc-bc}$ of all traces on the stochastic graph of the sampling experiment $d^3_{cc-bc}$. The function $p^*_{cc-bc}$ assigning the weight to each trace from the set $\Omega^*_{cc-bc}$ is the probabilistic distribution on the set $\Omega^*_{cc-bc}$. The pair $(\Omega^*_{cc-bc}, p^*_{cc-bc})$ is the probabilistic space called the space induced by the stochastic graph.

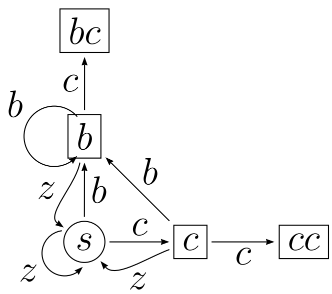


Fig. 5

The probabilistic spaces $(\Omega^*_{cc-bc}, p^*_{cc-bc})$ and $(\Omega^3_{cc-bc}, p^3_{cc-bc})$ are isomorphic (see [24], page 32). Therefore we assume the probabilistic space

$(\Omega^*_{cc-bc}, p^*_{cc-bc})$ as a probabilistic model of sampling experiment $d^3_{cc-bc}$. The following deductions and calculations are carried out in such probabilistic space.

In the probabilistic space $(\Omega^*_{cc-bc}, p^*_{cc-bc})$ the event $\{\ldots\alpha\}$ is a set of all traces leading to the node $\boxed{\alpha}$ on the graph in the Figure 5, $P(\ldots\alpha)$ is the sum of all weights of traces leading to the node $\boxed{\alpha}$ $(\alpha \in \{cc, bc\})$.

Let $A$ be a set of traces leading to the node $\boxed{bc}$, that do not contain any cycle. By $w(A)$ we denote the weight of traces belonging to the set $A$. Let $B$ be a set of traces leading to the node $\boxed{bc}$, that contain at least one cycle. By $w(B)$ we denote the weight of traces belonging to the set $B$. Here we have:

$$\{\ldots bc\} = A \cup B \quad \text{and} \quad A \cap B = \emptyset.$$

Because $P(\ldots bc)$ is the sum of weights of traces leading to the node $\boxed{bc}$, therefore:

$$P(\ldots bc) = P(A) + P(B) = w(A) + w(B). \tag{6.0.1}$$

Let $C$ denote a set of stochastic graph cycles of the sampling experiment $d_3^{cc-bc}$ containing the starting node start. Here we have:

$C = \{s \to s, s \to b \to s, s \to c \to s, s \to c \to b \to s, s \to b \to b \to s, s \to b \to b \to b \to s, \ldots, s \to c \to b \to b \to s, s \to c \to b \to b \to b \to s, \ldots\}$.

$C$ is an infinite but countable set, therefore we can enumerate its elements. Let $C = \{c_1, c_2, \ldots, c_j, \ldots\}$. Let $c_n$ be any fixed cycle and let $w(c_n)$ be its weight. Now let us consider a set $B_{c_n}$ made up of those traces leading to the node $\boxed{bc}$, that have $r(c_n)$ of the beginning edges creating the cycle $c_n$. Let $w(B_{c_n})$ denotes a sum of weights of all traces belonging to the set $B_{c_n}$. The set $B_{c_n}$ is equinumerous to the set $\{\ldots bc\}$. Let $x = P(\ldots bc)$. Therefore:

$$w(B_{c_n}) = w(c_n) \cdot x. \tag{6.0.2}$$

From

$$\bigcup_{c_n \in C} B_{c_n} = B \qquad \text{and} \qquad \forall c_j, c_k \in C \;\; [B_{c_j} \cap B_{c_k} = \emptyset], \tag{6.0.3}$$


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it follows that

$$P(B) = \sum_{c_n \in C} w(B_{c_n}). \tag{6.0.4}$$

From the equations (6.0.1) and (6.0.4) it results that

$$P(\dots bc) = w(A) + \sum_{c_n \in C} w(B_{c_n}).$$

If we take into consideration equation (6.0.2) in the above equation, then we have

$$x = w(A) + \sum_{c_n \in C} w(c_n) \cdot x = w(A) + x \cdot \sum_{c_n \in C} w(c_n).$$

So we get the equation

$$x = w(A) + x \cdot \sum_{c_n \in C} w(c_n).$$

After some transformation we have

$$x = \frac{w(A)}{1 - \sum_{c_n \in C} w(c_n)}. \tag{6.0.5}$$

Because $\sum_{c_n \in C} w(c_n) = \frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} + (\frac{1}{27} + \frac{1}{81} + \cdots) + (\frac{1}{81} + \frac{1}{243} + \cdots)$ and $w(A) = \frac{1}{9}$, therefore

$$P(\dots bc) = \frac{\frac{1}{9}}{1 - (\frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} + (\frac{1}{27} + \frac{1}{81} + \cdots) + (\frac{1}{81} + \frac{1}{243} + \cdots))}.$$

Finally we get: $P(\dots bc) = \frac{2}{3}$ and $P(\dots cc) = 1 - P(\dots bc) = \frac{1}{3}$.

The method presented in this work allows for probability calculation of the events type $\{\dots \alpha_j\}$ in case the graph of the sampling experiment $d^{u_1,\dots,u_n}_{\alpha_1,\dots,\alpha_k}$ has cycles with node start being one of inner nodes. This method enables us to find the probability of the event type $\{\dots \alpha_j\}$ on the basis of sampling experiment stochastic graph. Thereby this method may be used in case of any sampling experiment being the Markov chain and satisfying the above assumption.

The problem of drawing up a method of calculating the probability of the events type $\{\dots \alpha_j\}$ in case of any sampling experiment $d^{u_1,\dots,u_n}_{\alpha_1,\dots,\alpha_k}$ is still open.

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